Public Debt under Excusable Default:
Why do Governments Borrow so Much?

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Abstract

The present paper computes the debt-to-GDP ratio chosen by a self-interested government that engages in 'excusable' default. The government maximizes the utility of its own consumption over a period of time that is the government's expected stay in office, which the government fearing loss of office upon default seeks to extend to the fullest by defaulting only 'excusably' when unable to muster the funds necessary for debt service. The chosen debt ratio appears to be much closer to prevailing ratios than that chosen under the alternative assumption of an altruistic government engaging in strategic default: our baseline case has debt-to-GDP ratio 82% under excusable default, far above the 2.7% under strategic default.

1 Introduction

Recent years have seen a dramatic increase in many OECD countries' public debt. Our purpose in the present paper is to develop and calibrate a model that can at least partially reproduce the high levels of public debt now prevailing in a large number of developed countries.

Our model in its baseline calibration has debt-to-GDP ratio 82%. We are able to obtain so high a debt ratio because we depart from existing work in one central dimension: where previous work has assumed that governments default strategically, weighting the costs of debt service against those of default in their decision whether to service their debt or to default, our work assumes that governments engage in what Grossman and Van Huyck (1988) call excusable default. Excusable default has a government default only when the entirety of the resources it can muster, the country's maximum primary surplus and any proceeds from new debt issuance, fail to cover the cost of debt service. Thus, whereas strategic default can be viewed as a matter of will (the government decides to default as the result of a cost-benefit analysis that deems default more attractive than debt service), excusable default can be viewed as a matter of means (the government cannot but default as it lacks the means fully to service its debt). Collard, Habib, and Rochet (2015) have used the concept of excusable default to compute the maximum level of debt a government can sustain; we use that same concept to compute the debt level a government chooses. The debt levels we compute are much closer to prevailing debt levels than are those computed under strategic default: the same baseline calibration that has debt-to-GDP ratio 82% under excusable default has ratio 2.7% under strategic default.

The central intuition of our result is simple: the debt level chosen by a government depends, inter alia, on the cost of debt; lenders can be expected to be much more willing to provide high levels of debt at reasonable interest rates when they expect borrowers to do their utmost to service that debt than when they expect borrowers continuously to trade off the costs and benefits of debt service and default in deciding whether to service the debt. Debt being cheaper under excusable default, the government optimally chooses to borrow more in that case. Indeed, we find optimal debt to be close to maximum sustainable debt: the former but not the latter accounts for the government's concern for future payoffs; these are jeopardized only little through debt-induced default, because the low volatility of growth makes the probability of default very low even at maximum sustainable debt; there is therefore little reason for the government to choose optimal debt much below maximum sustainable debt, which represents the most advantageous trade-off between payment promised and proceeds received.[1]

Underlying the central assumption of excusable default is that of a self-interested government: the government does its utmost to stave off default because it expects to lose power upon default; senior members of a government that has lost power see the end of their political careers.[2] A self-interested government that engages in excusable default therefore maximizes not total consumption but that which accrues to the government and its favored constituents, over a

[1]The adjective 'optimal' denotes optimality from the perspective of the government. As the next paragraph makes clear, optimal debt is not necessarily optimal from the perspective of the country.

[2]Section 2 further justifies the assumption of excusable default.

period of time that extends not over an infinite horizon but over the government's expected time in office. This is in contrast to governments that engage in strategic default, which are generally assumed to behave altruistically, maximizing the entire population's consumption over the infinite horizon that spans the successive lifetimes of the country's present and future generations.[3] An implication of the distinction between self-interested and altruistic government is that the former is concerned with total public debt, both external and domestic, whereas the latter is concerned only with external public debt, domestic public debt being but a transfer between nationals.[4]

We examine the sensitivity of optimal debt and its associated default probability (PD) to various parameters of interest: the maximum primary surplus (MPS) which, along with the proceeds from new debt issuance, serves to service maturing debt, the fraction of total output that accrues to the self-interested government, the government's risk aversion, its probability of remaining in power, and its discount factor, the risk-free interest rate, and the mean and volatility of the rate of growth in output. Changes in the maximum primary surplus, the interest rate, and the mean and volatility of the growth rate have major impacts on optimal debt. For example, optimal debt increases from 40% of GDP to 160% as MPS increases from 2.5 to 10%. These changes appear to be due to changes in maximum sustainable debt (MSD): optimal debt is generally only a few percentage points below maximum debt; large changes in MSD consequently result in large changes in optimal debt. MSD increases in MPS and in the mean growth rate, reflecting the increased availability of resources for debt service; it decreases in the risk-free interest rate and in the volatility of the growth rate, the former result reflecting the increased attractiveness of the risk-free investment opportunity and the latter the decreased attractiveness of its now riskier alternative. In contrast to the large changes in optimal debt, the probability of default is hardly if at all affected by changes in the four parameters: changes in these parameters are accommodated nearly exclusively through changes in the quantity (level) rather than the quality (risk) of debt. Collard et al. (2015) find a similar result for MSD, which they attribute to their – and our – assumptions of lognormally distributed growth rate and zero recovery in default.

Confirming the central role of maximum debt for optimal debt, there is little change in optimal debt where MSD is left unaffected. Optimal debt decreases by only a few percentage points as the government's take or its probability of remaining in power increase, or as its discount factor decreases: the larger stake in the future these imply leads the government to decrease the probability of default by decreasing indebtedness. Much the same is true of risk-aversion: a more risk-averse government decreases debt in order to decrease the probability of default.

In order to shed further light on our results and on the importance of our assumption of excusable default, we consider the alternative case of an altruistic government that engages

[3] Amador (2004), Cuadra and Sapriza (2008), Hatchondo, Martinez, and Sapriza (2009), Amador and Aguiar (2011), and Acharya and Rajan (2013) are exceptions in this regard.

[4] Reinhart and Rogoff (2011) stress the importance of domestic public debt.

in strategic default. We adapt our model to that case, using Aguiar and Gopinath's (2006) 2% loss of output in default and Arellano's (2008) 73.4% probability of escaping default.[5] We obtain optimal debt ratio 2.7%, slightly above Arellano's 1% and below Aguiar and Gopinath 5%, and very much below the 82% ratio obtained under excusable default.[6] While the 2.7% and 82% ratios are not strictly comparable, the former pertaining to external public debt and the latter to total public debt, the difference is so large as to provide at least partial support for our assumptions of excusable default and self-interested government. We note that it is possible to obtain as high a value of optimal debt with strategic default as with excusable default, but that this implies unreasonably high costs of default (48.25% of GDP), or unreasonably low probability of escaping default (2.2% per annum).

Our baseline case has probability of default at optimal debt 0.106%, which corresponds to the rather improbable frequency of one default per millennium. Introducing the possibility of growth collapses (Rietz, 1988; Barro, 2006; Barro and Ursua, 2011) increases the probability of default at optimal debt to 0.953%, a much more reasonable one default per century; optimal debt decreases to 68%.

The paper proceeds as follows. Section 2 briefly reviews the literature. Section 3 derives the expression for maximum sustainable debt. Sections 4 and 5 derive the Bellman equations for optimal debt under excusable and strategic default, respectively. Section 6 parametrizes the model. Section 7 computes optimal debt under excusable default and examines its sensitivity to parameter values. Section 8 computes optimal debt under strategic default, analyzes its sensitivity to parameter values, and compares it to optimal debt under excusable default. Section 9 introduces the possibility of growth collapses. Section 10 computes for the United States the values initially computed for the Euro Area. Finally, Section 11 concludes.

2 Literature Review

The extensive literature on sovereign debt is a testimony to the importance of that topic.[7] Our paper is in the line of two strands of work within that literature, the first on optimal sovereign debt and the second on maximum sustainable debt. The work on optimal sovereign debt has quantified, refined, and extended Eaton and Gersovitz's (1981) seminal contribution; it has generally maintained their assumption of strategic default. Collard et al. (2015, pp. 386-387) briefly review various refinements to Eaton and Gersovitz, from Aguiar and Gopinath's (2006) incorporation of a trend into the output process, through Arellano's (2008) asymmetric cost of default, Mendoza and Yue's (2012) endogenous cost of default, Cuadra and Sapriza's (2008) political risk, Yue (2010) and Benjamin and Wright's (2009) renegotiation in default, Hatchondo and Martinez (2009) and Chatterjee and Eyigungor's (2012) debt maturity, and Fink and Scholl's

[5] Arellano's (2008) 28.2% quarterly probability of escaping default is our annual 73.4%.

[6] The values for Aguiar and Gopinath (2006) and Arellano (2008) are those reported in Cohen and Villemot (2013, Table 1).

[7] See the surveys by Aguiar and Amador (2014) and Panizza, Sturzenegger, and Zettelmeyer (2009), the monographs by Reinhart and Rogoff (2009) and Sturzenegger and Zettelmeyer (2006), and the references therein.

(2015) conditionality, to Cohen and Villemot's (2013) 'prepaid' cost of default.[8] Cohen and Villemot (Table 1) report the corresponding optimal debt ratios. These range from 1% (Arellano, 2008) to 38% (Cohen and Villemot, 2013) of GDP; they generally represent external public debt only, not the total public debt which we seek to reproduce.

The work on maximum sustainable debt has received much of its impetus from the aforementioned increase in OECD country debt.[9] Bohn (1998, 2008) has analyzed the requirements for sustainability, which Gosh, Kim, Mendoza, Ostry, and Qureshi (2013) have used to develop of measure of maximum debt and the 'fiscal space' it affords. Tanner (2013) has developed a measure of maximum liability that is more equity- than debt-like. As do Gosh et al., Collard et al. (2015) develop a debt-like measure of maximum sustainable debt. Collard et al.'s measure is perhaps 'less maximum' and 'more sustainable' than Gosh et al.'s, in the sense that any shortfall in growth below that necessary to service maximum debt implies certain default for Gosh et al., whereas it implies more probable but still uncertain default for Collard et al.

Our paper's concern is with optimal debt, which is computed as being some debt level short of maximum sustainable debt. Optimal debt falls short of maximum debt because of government's concern with – its own – future welfare: higher debt implies higher probability of default; default implies foregone utility; a government concerned with future welfare therefore seeks to avoid default by choosing a level of debt lower than maximum debt.

As noted previously, the assumption of excusable default is central to our analysis. How realistic is it? Very! Levy Yeyati and Panizza (2011) provide strong evidence of governments' reluctance to default: governments appear to default only as a last resort, after they have tried every possible way of staving default off. While debt service is costly, default is generally even costlier, especially from the point of view of a government that can generally expect to lose power in the aftermath of default (Borensztein and Panizza, 2009; Malone, 2011). Even a less than fully self-interested government may do its utmost to avoid default: Tomz (2007) has argued that creditors are much more lenient towards borrowers for whom default was clearly unavoidable than those who are perceived to have been too quick to default; Bolton and Jeanne (2011) have noted the potential of sovereign default to jeopardize the proper functioning of an entire banking system, in view of government bonds' importance as collateral for bank loans.

3 Maximum Sustainable Debt

The first step in our analysis consists in estimating maximum sustainable debt. We follow Collard et al. (2015) for that purpose.

Let y_t denote a given country's output in period t, D_t the debt raised by the country's government in that same period, to be repaid in its entirety in the following period $t+1$ (we assume that debt is fully amortized every period), B_t the proceeds obtained by the government

[8] See also Section 6 in Aguiar and Amador (2014). Hamann (2002) constitutes an early attempt at calibrating optimal sovereign debt levels.

[9] Such work can, however, be traced to Aaron's (1966) early work on constant debt ratios, as well as Reinhart, Rogoff, and Savastano's (2003) later work on 'debt intolerance.'

in period t from raising that debt (we assume all debt is issued in the form of zero-coupon bonds), αy_t the maximum primary surplus (MPS) the country can achieve on a sustainable basis, and r the risk-free interest rate.[10] Expressed as a fraction of period-t output y_t, debt and proceeds can be written $d_t = D_t/y_t$ and $b_t = B_t/y_t$. Let $g_{t+1} \equiv y_{t+1}/y_t$ denote the rate of growth in output between periods t and $t+1$; g is assumed to be i.i.d. over the range $[0, \infty)$; we denote $F(.)$ and $f(.)$ the cdf and pdf of g , respectively.

We seek maximum sustainable debt (MSD) d_M and maximum sustainable borrowing (MSB) b_M.[11] We start with the latter. If the country were to raise debt $d_t y_t$ in period t, it would default on that debt in the period $t+1$ in which the debt is due when

$$\alpha y_{t+1} + b_M y_{t+1} < d_t y_t. \tag{1}$$

The RHS represents the debt to be repaid in period $t+1$, the LHS the resources available to the government for that purpose; these are the sum of the MPS the country can achieve, αy_{t+1}, and the maximum proceeds from sustainable new borrowing in period $t+1$, $b_M y_{t+1}$.

Rearranging (1), default occurs when the growth rate g_{t+1} is such that

$$g_{t+1} < \frac{d_t}{\alpha + b_M} \equiv g_{E,t+1}; \tag{2}$$

$g_{E,t+1}$ denotes the critical rate necessary to avoid default. Assuming zero recovery in default, borrowing proceeds $b_t y_t$ corresponding to debt issued $d_t y_t$ equal

$$b_t y_t = \frac{d_t y_t}{1+r} \Pr\left[g_{t+1} \geqslant g_{E,t+1}\right] = \frac{d_t y_t}{1+r}\left[1 - F\left(g_{E,t+1}\right)\right]. \tag{3}$$

Using (2) to write

$$d_t = \left(\alpha + b_M\right) g_{E,t+1} \tag{4}$$

and dividing (3) by y_t, we can write borrowing proceeds as a fraction of output

$$b_t = \frac{\alpha + b_M}{1+r} g_{E,t+1}\left[1 - F\left(g_{E,t+1}\right)\right]. \tag{5}$$

Borrowing proceeds b_t display a 'Laffer Curve' property in the critical rate necessary to avoid default g_{Et+1}. Proceeds are zero when that rate is zero, as only when no debt is raised can there be no default when growth and consequently output are zero ($g_{t+1} = g_{E,t+1} = 0$ implies $y_{t+1} = 0$).[12] Proceeds are also zero when that rate is infinite, as default occurs with certainty in such case. Proceeds increase and then decrease between these two extremes.[13] Clearly, then, borrowing proceeds are maximized when the critical rate maximizes $g\left[1 - F(g)\right]$; that rate does not depend on t because the cdf $F(.)$ does not. These maximum proceeds are sustainable,

[10]Maximum primary surplus is the difference between maximum government revenues – excluding the proceeds from new debt issuance – and minimum government spending – excluding the cost of debt service. Minimum government spending is not zero: some basic level of essential services must be provided in all circumstances (e.g., law and order, health, education, etc. . .).

[11]MSD d_M and MSB b_M will be seen below not to depend on date t.

[12]This is perhaps too literal an interpretation of (5). The basic idea is that debt must be very low for default to be avoided even for very low growth and output realizations.

[13]Note that $\partial b_t / \partial g_{E,t+1}|_{g_{E,t+1}=0} > 0$.

for they rely for debt service on future proceeds that are themselves sustainable. Maximum sustainable borrowing b_M therefore is the fixed point

$$b_M = \frac{\alpha + b_M}{1+r} g_M \left[1 - F\left(g_M\right)\right] \tag{6}$$

$$\Rightarrow b_M = \frac{\alpha g_M \left[1 - F\left(g_M\right)\right]}{1 + r - g_M \left[1 - F\left(g_M\right)\right]} \tag{7}$$

where

$$g_M = \arg\max_{g} \ g\left[1 - F\left(g\right)\right]. \tag{8}$$

Note that the fixed point property precludes reliance on unbounded borrowing ratios.

Using (4), we have that MSD d_M equal

$$d_M = \left(\alpha + b_M\right) g_M = \frac{\alpha\left(1+r\right) g_M}{1 + r - g_M \left[1 - F\left(g_M\right)\right]}. \tag{9}$$

We denote PD_M the corresponding probability of default, $PD_M \equiv F\left(g_M\right)$.

4 Optimal Debt under Excusable Default

We now turn to the determination of optimal debt, making use of our previous analysis of maximum debt and borrowing for that purpose.

The self-interested government's period-t consumption is $\varphi y_t + b_t y_t - d_{t-1} y_{t-1}$, where φ, $\varphi < 1$, denotes the fraction of output that accrues to the government, which further makes use of the entirety of net debt proceeds – new debt proceeds minus debt repayment. Note that the self-interested government does not distinguish between foreign and domestic debt, as both constitute sources of additional funds to the government as well as claims on funds that would otherwise be available to the government; debt under excusable default therefore should be viewed as total debt.

We denote $u\left(.\right)$ the utility function that the government maximizes, β, $\beta \leqslant 1$, the discount factor, and θ, $\theta \leqslant 1$, the probability that the government remain in power: a self-interested government's horizon coincides with the government's expected time in office. We assume that the senior members of a government that has defaulted never again return to power after default; their political lives end with default, with corresponding payoffs zero.

The value function $V_E\left(d_{t-1} y_{t-1}, y_t\right)$ in period t is

$$V_E\left(d_{t-1} y_{t-1}, y_t\right) = \max_{d_t} \ u\left(\varphi y_t + b_t y_t - d_{t-1} y_{t-1}\right) + \theta\beta\mathbb{E}\left[V_E\left(d_t y_t, y_{t+1}\right)\right], \tag{10}$$

where the expectation is over period-$t+1$ output. The government chooses debt issuance d_t and corresponding proceeds b_t to maximize the value function; (3) relates d_t and b_t.

Assuming CRRA utility $u\left(c\right) = c^{1-\gamma}/\left(1-\gamma\right)$, we can rewrite the value function as

$$V_E\left(d_{t-1} y_{t-1}, y_t\right) = V_E\left(\frac{d_{t-1} y_{t-1}}{y_t}, 1\right) y_t^{1-\gamma} = v_E\left(\omega_t\right) y_t^{1-\gamma}, \tag{11}$$

where $\omega_t \equiv d_{t-1} y_{t-1}/y_t$ denotes the stock of debt carried over from period $t-1$ into period t, expressed not as a fraction of period-$t-1$ output as is d_{t-1} but as a fraction of period-t output.

This is because it is out of period-t output and not out of period-$t-1$ that the debt $d_{t-1}y_{t-1}$ must be repaid. It is therefore $\omega_t y_t$ and not $d_{t-1}y_{t-1}$ that matters for the determination of default in period t. Intuitively, even very high debt carried over from the previous period can be serviced if growth between that period and the present has been very high. We refer to ω as the realized debt ratio, in contrast to the promised debt ratio d. In a manner analogous to these two ratios, $v_E(.)$ expresses the value function as a fraction of output; it decreases in realized debt ω_t raised in the previous period, to be repaid in the present.[14]

Using (10) and (11), we can write

$$\begin{aligned} v_E(\omega_t)\, y_t^{1-\gamma} &= \max_{d_t}\; u(\varphi + b_t - \omega_t)\, y_t^{1-\gamma} + \theta\beta\mathbb{E}\left[v_E(\omega_{t+1})\, y_{t+1}^{1-\gamma}\right] \\ \Leftrightarrow v_E(\omega_t) &= \max_{d_t}\; u(\varphi + b_t - \omega_t) + \theta\beta\mathbb{E}\left[v_E(\omega_{t+1})\, g_{t+1}^{1-\gamma}\right], \end{aligned} \tag{12}$$

where $\omega_{t+1} \equiv d_t y_t / y_{t+1}$.

Substituting (4) and (5) into (12), we can write

$$\begin{aligned} v_E(\omega_t) = \max_{g_{E,t+1}}\; & u\left(\varphi + \frac{\alpha + b_M}{1+r} g_{E,t+1}\left[1 - F(g_{E,t+1})\right] - \omega_t\right) \\ & + \theta\beta \int_{g_{E,t+1}}^{\infty} v_E\left((\alpha + b_M)\frac{g_{E,t+1}}{g}\right) g^{1-\gamma} dF(g), \end{aligned} \tag{13}$$

where we have used of both the i.i.d property of the growth rate and the fact that the payoff remains zero after default, a consequence of the government's loss of power following default and its (senior) members' ensuing retirement from politics.[15]

Proposition 1 *Under the assumption $\beta\theta E(g^{1-\gamma}) < 1$, the Bellman equation (13) satisfies the Blackwell conditions for a unique solution $g_E^\star(\omega)$. Given realized debt ratio ω, optimal debt $d_E^\star(\omega)$, borrowing proceeds $b_E^\star(\omega)$, and default probability $PD_E^\star(\omega)$ are*

$$d_E^\star(\omega) \equiv (\alpha + b_M)\, g_E^\star(\omega), \tag{14}$$

$$b_E^\star(\omega) \equiv \frac{\alpha + b_M}{1+r} g_E^\star(\omega)\left[1 - F(g_E^\star(\omega))\right], \tag{15}$$

$$PD_E^\star(\omega) \equiv F(g_E^\star(\omega)). \tag{16}$$

A comparison of (5) and (13) suggests that $d_E^\star(\omega) = d_M$, $b_E^\star(\omega) = b_M$, and $PD_E^\star(\omega) = PD_M$, $\forall\omega$, if $\theta = 0$.[16] We show in Section 7.2 that this is indeed the case.

[14] That $v_E'(.) < 0$ is immediate from (12) below.

[15] Specifically, we use

$$\begin{aligned} \mathbb{E}\left[v_E(\omega_{t+1})\, g_{t+1}^{1-\gamma}\right] &= \mathbb{E}\left[v_E\left(\frac{d_t}{g_{t+1}}\right) g_{t+1}^{1-\gamma}\right] = \mathbb{E}\left[v_E\left(\frac{d_t}{g}\right) g^{1-\gamma}\right] \\ &= \mathbb{E}\left[v_E\left((\alpha + b_M)\frac{g_{E,t+1}}{g}\right) g^{1-\gamma}\right] = \int_{g_{E,t+1}}^{\infty} \left\{v_E\left((\alpha + b_M)\frac{g_{E,t+1}}{g}\right) g^{1-\gamma}\right\} dF(g). \end{aligned}$$

[16] The same is obviously true of $\beta = 0$, but this is not a realistic discount factor. In contrast, it is conceivable for a government to be so unpopular as to be near-certain of being voted out of power, $\theta = 0$.

5 Optimal Debt under Strategic Default

For comparison purposes, we compute optimal debt in the case of strategic default. We mainly follow Arellano (2008), which we modify slightly in order to exploit the growth rate's i.i.d. property.

We assume that a country whose government has defaulted strategically is excluded from international financial markets for at least one period. At the end of that period, the country escapes default and returns to financial markets with probability λ; with probability $1-\lambda$, the country remains in default for one additional period, at the end of which the 'escape' process repeats itself. While it is in default, the country loses a fraction τ of its output. Thus, if the country should be in default in period t in which output is y_t, the country's consumption would be $\varphi(1-\tau)y_t$; $\varphi=1$ for the altruistic government which maximizes the entire population's consumption: recall from our discussion in the Introduction that it is altruistic governments that engage in strategic default.[17] If in contrast the country should have access to financial markets in that period, consumption would be

$$\varphi y_t + B_t - D_{t-1} = \varphi y_t + b_t y_t - d_{t-1} y_{t-1}.$$

Note that debt represents only external debt, as domestic debt constitutes neither a source of funds nor a claim on funds for the country's population considered in its entirety.

We denote $V_D(y_t)$ the value function in default when output is y_t and $V_S(d_{t-1}y_{t-1}, y_t)$ outside default; a country that eschews default must service the debt $d_{t-1}y_{t-1}$ carried over from the previous period. We have

$$V_D(y_t) = u(\varphi(1-\tau)y_t) + \theta\beta\mathbb{E}\left[\lambda V_S(0, y_{t+1}) + (1-\lambda)V_D(y_{t+1})\right],$$

where we have assumed that a country that has defaulted repudiates all outstanding debt; $\theta=1$ for the altruistic government that maximizes the discounted lifetime utility of the country's present and future generations.

Using utility's CRRA form, $u(c) = c^{1-\gamma}/(1-\gamma)$ and the growth rate's i.i.d. property, we can write

$$\begin{aligned}
v_D y_t^{1-\gamma} &= \frac{(\varphi(1-\tau)y_t)^{1-\gamma}}{1-\gamma} + \theta\beta\mathbb{E}\left[\lambda v_S(0)\, y_{t+1}^{1-\gamma} + (1-\lambda)v_D y_{t+1}^{1-\gamma}\right] \\
\Leftrightarrow v_D &= \frac{\varphi^{1-\gamma}(1-\tau)^{1-\gamma}}{1-\gamma} + \theta\beta\mathbb{E}\left[\lambda v_S(0)\, g_{t+1}^{1-\gamma} + (1-\lambda)v_D g_{t+1}^{1-\gamma}\right] \\
\Leftrightarrow v_D &= \frac{\frac{\varphi^{1-\gamma}(1-\tau)^{1-\gamma}}{1-\gamma} + \frac{\theta\lambda v_S(0)}{1+r}\mathbb{E}\left[g^{1-\gamma}\right]}{1 - \frac{\theta(1-\lambda)}{1+r}\mathbb{E}\left[g^{1-\gamma}\right]}. \qquad (17)
\end{aligned}$$

The value function outside default is

$$V_S(d_{t-1}y_{t-1}, y_t) = \max\left\{V_D(y_t), \max_{d_t} u(\varphi y_t + b_t y_t - d_{t-1}y_{t-1}) + \theta\beta\mathbb{E}\left[V_S(d_t y_t, y_{t+1})\right]\right\}.$$

[17] We include φ despite it being equal to one for comparison with the analysis of Section 4; we do likewise for θ below.

Making use of the CRRA assumption and defining $\omega_t \equiv (d_{t-1}y_{t-1})/y_t$, we can write

$$v_S(\omega_t) = \max\left\{v_D, \max_{d_t} u(\varphi + b_t - \omega_t) + \theta\beta\mathbb{E}\left[v_S(\omega_{t+1})\, g_{t+1}^{1-\gamma}\right]\right\}. \tag{18}$$

As in the case of excusable default, we need to determine the relation between the face value of zero coupon debt raised in period t and due in period $t+1$, $d_t y_t$, and its corresponding proceeds in period t, $b_t y_t$. For that purpose, we need to determine the range of realized debt ratios for which the government chooses to service its debt in period $t+1$, that is, the range of debt ratios ω_{t+1} such that $v_S(\omega_{t+1}) \geqslant v_D$. We define ω_S to be the maximum such ratio, $v_S(\omega_S) = v_D$; ω_S does not depend on t because of the i.i.d. distribution of the growth rate g; it is a maximum because $v_S'(.) < 0$ from (18). We refer to ω_S as maximum feasible debt (MFD). Default occurs over the range of debt ratios $\omega_{t+1} > \omega_S$, that is, over the range of growth rates such that

$$\frac{d_t}{g_{t+1}} > \omega_S$$

$$\Leftrightarrow g_{t+1} < \frac{d_t}{\omega_S} \equiv g_{S,t+1}. \tag{19}$$

We can therefore write

$$d_t = \omega_S g_{S,t+1} \tag{20}$$

and

$$b_t = \frac{d_t}{1+r}\Pr\left[g_{t+1} \geqslant g_{S,t+1}\right] = \frac{\omega_S}{1+r} g_{S,t+1}\left[1 - F(g_{S,t+1})\right]. \tag{21}$$

Substituting (20) and (21) into (18), we can write

$$\begin{aligned} v_S(\omega_t) = \max\Bigg\{ v_D, \max_{g_{S,t+1}}\ & u\left(\varphi + \frac{\omega_S}{1+r} g_{S,t+1}\left[1 - F(g_{S,t+1})\right] - \omega_t\right) \\ & + \theta\beta\left[\int_0^{g_{S,t+1}} v_D g^{1-\gamma} dF(g) + \int_{g_{S,t+1}}^{\infty} v_S\left(\frac{\omega_S g_{S,t+1}}{g}\right) g^{1-\gamma} dF(g)\right]\Bigg\}. \end{aligned} \tag{22}$$

A comparison of (13) and (22) that abstracts from the value of default v_D reveals the close resemblance between the two maximizations: the latter has ω_S replace $\alpha + b_M$ in the former. This suggests that any difference between *optimal* values we obtain will be due in the first instance to the difference between *maximal* values under excusable and strategic default. We show in Section 8.1 that this is indeed the case.

We solve the Bellman equation (22) for $g_S^\star(\omega)$. Given realized debt ratio ω, optimal debt $d_S^\star(\omega)$, borrowing proceeds $b_S^\star(\omega)$, and default probability $PD_S^\star(\omega)$ are

$$d_S^\star(\omega) \equiv \omega_S g_S^\star(\omega), \tag{23}$$

$$b_S^\star(\omega) \equiv \frac{\omega_S}{1+r} g_S^\star(\omega)\left[1 - F(g_S^\star(\omega))\right], \tag{24}$$

$$PD_S^\star(\omega) \equiv F(g_S^\star(\omega)). \tag{25}$$

6 Parametrization

We parametrize our model using Euro Area data. We choose the Euro Area over the United States, the United Kingdom, or Japan because our model assumes no central bank intervention in government debt markets; this assumption is arguably more true of the European Central Bank than it is of the Federal Reserve, the Bank of England, or the Bank of Japan.[18] We assume the output growth process to be log-normally distributed. We set its mean and standard deviation to match those of Euro Area per-capital output over the period 1990-2013, $\mu = 1.02\%$ and $\sigma = 2.12\%$; per-capita output is obtained from the (Euro) Area Wide Model Database. We set the interest rate r equal to the real government bond yield for the Euro Area over that same period, $r = 1.04\%$; it is obtained from the IMF's International Financial Statistics. In accordance with IMF (2011) estimates, we set the maximum primary surplus equal to 5% of GDP; $\alpha = 0.05$. We set the fraction of output that is of concern to the self-interested government that engages in excusable default $\varphi = 0.5$; this is somewhat higher than the ratio of government spending to GDP because the government may be concerned with part of private spending – that by its favored constituents for example – in addition to public spending; we set $\varphi = 1$ for the altruistic government that engages in strategic default. We set $\theta = 0.6$ in the case of excusable default and $\theta = 1$ in that of strategic default; in the former case, the probability of remaining in power θ concedes a moderate advantage to the government over the opposition. We follow Arellano (2008) in setting the probability of escaping default to 73.4%; $\lambda = 0.734$.[19] We follow Aguiar and Gopinath (2006) in setting the loss of output in default equal to 2% of GDP; $\tau = 0.02$. In accordance with much of the growth literature, we set the discount factor $\beta = 0.95$. Finally, we set the CRRA coefficient $\gamma = 0.5$.[20] Table 1 shows the various parameter values, distinguishing between the two cases of excusable and strategic default.

We use a value-iteration procedure to solve the Bellman equations (13) and (22). We then use the decision rules (14)-(16) and (23)-(25) to generate through simulation three time series – debt-to-GDP, borrowing proceeds-to-GDP, and default probability – for each of the two cases of excusable and strategic default. We compute the average values of each of these times series, repeat the procedure 1,000 times, and average the results again to obtain the averages $d_E^\star$, $b_E^\star$, and PD_E^* for excusable default and v_D, ω_S, $d_S^\star$, $b_S^\star$, and PD_S^* for strategic.[21]

[18]Note that we do not need to assume a single government choosing public debt for the entire Euro Area. Our analysis can be viewed as averaging the choices made by the different governments of the individual Euro Area countries.

[19]As noted in Footnote 5, 0.734 is the annual equivalent to Arellano's (2008) quarterly 0.282.

[20]Our CRRA coefficient is bounded above by 1. To see why this is the case, note that the senior members of a government than has engaged in excusable default have zero payoff, as they are assumed never to return to power. A γ larger than 1 would result in the paradoxical situation in which governments would consistently be better off in default, for the zero payoff of default would then be higher than the negative payoff of debt service ($c^{1-\gamma}/(1-\gamma) < 0$ if $c > 0$ and $\gamma > 1$).

[21]No simulation is needed to compute d_M, b_M, or PD_M.

7 Results: Excusable Default

7.1 Basic Results

Table 2 shows the results for excusable default. Our main interest is in the results in the first row, which use the parameter values for excusable default ($\varphi = 0.5$ and $\theta = 0.6$). Maximum sustainable debt d_M is 83.2% of GDP, maximum sustainable borrowing b_M is 81.8%. The very small difference between MSD and MSB is due to the low risk-free interest rate, $r = 1.04\%$, and the very low probability of default at MSD, $PD_M = 0.765\%$. This very low probability in turn reflects the low volatility of growth, $\sigma = 2.12\%$, which implies an extremely steep transition from near-zero to near-one probability of default PD as a function of the face value of debt d (see Figure 1). Any face value of debt other than one associated with a very low probability of default would therefore see a collapse in borrowing proceeds. This is confirmed by the Laffer curve in Figure 2, which shows a dramatic decline in borrowing proceeds b past MSD d_M.[22]

The very high MSD, at least in comparison to the values generally obtained under strategic default, reflects the government's high debt service capacity, which under excusable default constitutes the primary determinant of maximum sustainable debt: there can be no excusable default when the government has the capacity to service debt. A government's debt service capacity in turn depends on the maximum primary surplus, the risk-free interest rate, the growth rate, and the ability repeatedly to raise new debt, which in the absence of default effectively serves to make all future primary surpluses available for the repayment even of debt of maturity only one year. As the probability of default is very low at MSD ($PD_M = 0.765\%$), new debt can be counted upon with near certainty, and MSD is raised far above MPS ($d_M = 83.2\% \gg 5\%$ =MPS). We examine the sensitivity of MSD to the maximum primary surplus, the risk-free interest rate, and the growth rate below.

Optimal sovereign debt $d^\star_E$ equals 82.1% of GDP; it is extremely close to MSD d_M. To understand this result, recall from Section 3 that MSD is the level of debt that maximizes borrowing proceeds; it would therefore be chosen by governments concerned only with the current payoff, that is, governments for which $\theta = 0$. As such is not the case, a governments who wishes to avoid jeopardizing future payoffs through debt-induced default chooses a level of debt lower than MSD. This consideration fails to decrease optimal debt markedly below maximal, however, because the probability of default PD_M at MSD is very small already: even a level of debt very close to MSD has very low probability of default. The government consequently chooses optimal debt close to MSD, which represents the most advantageous trade-off between payment promised and proceeds received, the apex of the Laffer curve in Figure 2. The probability of default at optimal debt is extremely low, $PD^\star_E = 0.106\%$; it combines with the low risk-free interest rate to make optimal borrowing proceeds $b^\star_E$ very close to optimal debt $d^\star_E$, $b^\star_E = 81.2\%$ of GDP.

Recall that our model is calibrated to the Euro Area. Euro Area public debt now stands at 92% of GDP, close to but nonetheless larger than computed MSD at 83.2% and optimal debt at

[22]In order further to highlight the importance of low volatility to the steepness of the transitions in default probability and borrowing proceeds, Figures 1 and 2 show the case $\sigma = 21.2\%$ in addition to $\sigma = 2.12\%$.

82.1%.

The results in the second to fourth rows are obtained by replacing one or both parameter values for excusable default by those for strategic default ($\varphi = 1$ and $\theta = 1$).[23] They are of interest mainly in that optimal debt $d_E^\star$, proceeds $b_E^\star$, and probability of default $PD_E^\star$ are to be compared with the results in Section 8 computed under strategic default, obtained with the same parameter values. Clearly, MSD d_M, MSB b_M, and probability of default PD_M are unaffected by the changes in φ and θ, as maximum debt is computed independently of any concern for future payoffs.[24] In contrast, optimal debt $d_E^\star$, proceeds $b_E^\star$, and probability of default $PD_E^\star$ all decrease as compared to their earlier values, reflecting the now greater importance attached to future payoffs. This is immediate for θ, perhaps slightly less so for φ: the concavity of the utility function implies that φ and borrowing proceeds are strategic substitutes; an increase in φ therefore decreases optimal debt further away from maximum debt both by decreasing the benefits to be had from increased proceeds in the current period and by increasing the payoff to avoiding default in the next period.[25] Reassuringly, changes in values are largest when both parameter values are changed.[26]

7.2 Sensitivity Analysis

Figures 3a to 4d show the sensitivity of maximum and optimal debt and their corresponding proceeds and associated probabilities of default to the exogenous parameters. Figures 3a, 4a, 4b, and 4c confirm the importance of the MPS, the risk-free interest rate, and the mean and volatility of the growth rate to MSD. MSD increases from 40 to 160% of GDP as MPS α increases from 2.5 to 10% of GDP; it increases from 75 to 110% of GDP as the mean growth rate μ increases from 1.5 to 2.5%: a government that generates a higher primary surplus from a faster growing economy has more resources available for debt service; it can therefore borrow more.[27] MSD decreases from 90 to 65% of GDP as the risk-free interest rate varies over the same interval as μ: a higher opportunity cost of capital decreases lenders' willingness to lend to the government, which consequently can borrow less. MSD decreases from 115 to 70% of GDP as the volatility of the growth rate σ increases from 1.5 to 2.5%, the same interval as μ: the more volatile is growth, the greater the likelihood of low growth realizations that leave the government unable to service its debt, the less the government can borrow.

In line with our discussion above, optimal debt $d_E^\star$ closely tracks MSD d_M. This is a consequence of the very low probabilities of default at MSD, never above 1% over the ranges considered; the probabilities of default at optimal debt are lower still, rarely exceeding 0.2%. Note that PD_M is invariant in MPS α, the mean growth rate μ, and the risk-free interest rate r; this is an artifact of growth's lognormal distribution and zero recovery: changes in parameter

[23] The results in the last row will be discussed in Section 9.

[24] Formally, equations (7), (8), and (9) involve neither θ nor φ.

[25] See (12) and note that $v_E(\omega_{t+1})$ in $E\left[v_E(\omega_{t+1})\, g_{t+1}^{1-\gamma}\right]$ is itself an increasing function of φ.

[26] Changes in $d_E^\star$, $b_E^\star$, and $PD_E^\star$ are greater between the first row and the fourth (φ *and* θ changed) than between the first row and the second or third rows (φ *or* θ changed, respectively).

[27] The numbers reported are rounded for ease of readability.

values that should be accommodated in both the size of debt and its riskiness are accommodated only by the former, leaving the latter unchanged.[28] As expected, PD_M increases in volatility σ, reflecting the higher probability of default associated with more volatile growth. The probability of default $PD_E^\star$ generally follows optimal debt $d_E^\star$, increasing where $d_E^\star$ increases (α and very slightly μ) and decreasing where $d_E^\star$ decreases (very slightly r). That $PD_E^\star$ (very slightly) increases in volatility σ despite the decrease in $d_E^\star$ suggests that the direct effect of σ on $PD_E^\star$ dominates its indirect effect through the decreasing $d_E^\star$. Turning to borrowing proceeds b_M and $b_E^\star$, we note that these follow the same pattern as debt, from which they differ only very little by virtue of the low risk-free interest rate and probabilities of default.

In Figures 3b, 3c, 3d, and 4d, maximum debt d_M, proceeds b_M, and probability of default PD_M do not change. This is because the parameters φ, γ, θ, and β pertain to a trade-off between present and future payoffs that has no relevance for maximum debt. This trade-off is, however, central to optimal debt $d_E^\star$, which decreases in all four parameters: as noted in Section 7.1, an increase in φ or θ increases the relative importance of future payoffs, which are not to be jeopardized by default; the same is true of an increase in β; an increase in γ increases the desirability of smoothing consumption over time, with future consumption again not to be jeopardized through default. The probability of default $PD_E^\star$ is decreased by decreasing optimal debt $d_E^\star$, with attending decrease in optimal proceeds $b_E^\star$. Their diverging paths notwithstanding, maximum and optimal debt and borrowing remain close in value. Again, this is due to the very low probability of default at maximum debt: there is little need for optimal debt $d_E^\star$ markedly to deviate from maximum debt d_M for the probability of default at optimal debt $PD_E^\star$ to be very small. This can be seen in Figure 3d for example, where as θ increases from 0 to 1, the difference between d_M and $d_E^\star$ increases from zero to 3%, and the probability of default at optimal debt $PD_E^\star$ decreases to become effectively nil.[29] This suggests that optimal debt's deviation from the tracking of maximum debt is confined to a very narrow range.

8 Results: Strategic Default

8.1 Basic Results

Table 3 shows the results for strategic default. As in Section 7.1, our main interest is in the results in the first row, which use the parameter values for strategic default ($\varphi = 1$ and $\theta = 1$). The realized debt ratio beyond which the government defaults, maximum feasible debt MFD, is extremely small, $\omega_S = 2.876\%$ of GDP. This result is a consequence of the growth rate's very

[28] Formally, consider g_M and g in (8) and define $x_M \equiv [\log(g_M) - \mu]/\sigma$ and $x \equiv [\log(g) - \mu]/\sigma$. Use the lognormality of $F(.)$ to rewrite (8) as

$$\begin{aligned} x_M &= \arg\max_x \ \exp(\mu + \sigma x)[1 - \Phi(x)] \\ &= \arg\max_x \ \exp(\sigma x)[1 - \Phi(x)]. \end{aligned}$$

Clearly, $PD_M = \Phi(x_M)$ depends exclusively on σ.

[29] That $d_E^\star = d_M$, $b_E^\star = b_M$, and $PD_E^\star = PD_M$ at $\theta = 0$ in Figure 3d confirms the observation made after Proposition 1.

low volatility: as noted by Aguiar and Gopinath (2006) and Aguiar and Amador (2012), there is relatively little value to the insurance provided by borrowing when there is little volatility in output; there is therefore relatively little to restrain a government behaving strategically from defaulting; default occurs at low debt ratios. This is especially so under the present parametrization, because the proportional cost of default at $\tau = 2\%$ of GDP is low and the probability of escaping default at $\lambda = 73.4\%$ is high. Optimal debt $d_S^\star = 2.698\%$ of GDP is very close to ω_S, yet its associated probability of default, the probability that growth be less than $d_S^\star/\omega_S$, is very low, $PD_S^\star = 0.026\%$. Again, this is due to the very low volatility of growth; again, the low probability of default at optimal debt and the low interest rate combine to make optimal borrowing proceeds very close to optimal debt, $b_S^\star = 2.669\%$ of GDP. Optimal debt $d_S^\star$ is close to MFD ω_S for reasons similar to those discussed in Section 7, namely the desirability of maximum proceeds, mitigated only weakly by concern for future payoffs because of the low probability of default. That both forms of default see a very small difference between maximum (d_M, ω_S) and optimal ($d_E^\star$, $d_S^\star$) debt suggests that the difference between strategic and excusable default pertains not to the desirability of high debt levels but to their feasibility. This confirms the observation made towards the end of Section 5 that the difference between optimal debt under excusable and strategic default is first and foremost a consequence of the difference between maximal debt in these two cases. While part of the difference in optimal debt values must be attributable to the distinction between excusable default's total debt and strategic default's foreign debt, such distinction is very unlikely to account for the entire $d_E^\star - d_S^\star = 82.1\% - 2.7\% = 79.4\%$ difference: neither the Euro Area as as a whole nor individual Euro Area countries have 97% (= 79.4%/82.1%) of their public debt held by non-nationals.

The results in the second to fourth rows are obtained by replacing one or both parameter values for strategic default by those for excusable default ($\varphi = 0.5$ and $\theta = 0.6$). All values computed have the same order of magnitude as the values in the first row, with the exception of the probability of default at optimal debt $PD_S^\star$ in the case $\theta = 0.6$. This confirms, if there were the need to do so, that the order of magnitude difference between the values computed in Section 7 and 8 is due not to different parameter values but to the different assumptions regarding default.

Changes in φ and θ have two effects on debt and borrowing, one direct and the other indirect through the default payoff v_D in (17). The direct effect of the decrease in φ and θ is to decrease the importance attached future payoffs, thereby weakening the restraint on the government to engage in strategic default, possibly leading to a decline in maximum feasible debt and optimal debt and borrowing. The indirect effect through v_D is opposite, as the lower v_D that results from the decrease in φ and θ strengthens the restraint.[30] The decrease in debt and borrowing that results from the decrease in φ alone suggests that the direct effect dominates in that case; in contrast, the increase in debt and borrowing that results from the decrease in θ alone suggests that the indirect effect dominates in that other case. When both φ and θ decrease, the direct

[30]The last column of Table 3 shows the decrease in v_D that results from the decreases in φ and θ.

effect appears to dominate (just). Somewhat surprisingly, the offsetting effect of φ on θ does not extend from debt and borrowing to the probability of default at optimal debt: $PD_S^\star$ is the same in the third and fourth rows. This can be ascribed to the much greater sensitivity of $PD_S^\star$ to θ than to φ over the range of values considered, as will be seen in the sensitivity analysis of Section 8.2 below.

8.2 Sensitivity Analysis

Figures 5a to 6d show the sensitivity of maximum feasible debt ω_S, optimal debt $d_S^\star$, optimal borrowing proceeds $b_S^\star$, and the probability of default at optimal debt $PD_S^\star$ to the exogenous parameters. Figure 5a confirms our interpretation in Section 8.1 of the results in the third row of Table 3. As argued then, an increase in the fraction of output that is of concern to the government φ increases the importance attached future payoffs; it therefore serves to restrain the government from engaging in strategic default (direct effect), unless offset by the increase of v_D in φ (indirect effect). The direct effect dominates; it increases both ω_S and $d_S^\star$, the latter closely tracking the former. It also increases $b_S^\star$, but leaves $PD_S^\star$ essentially unchanged. This suggests that changes in the extent to which strategic default is restrained sometimes are accommodated along a single margin, the level of debt in the present case, rather than along the two possible margins that are the debt level and the default probability.

Figure 5b illustrates a case in which both the direct and indirect effects combine to weaken the restraint on the government to engage in strategic default. A larger probability of escaping default decreases the cost of default, thereby weakening the restraint on the government and decreasing maximum and optimal debt and borrowing. This direct effect is compounded by the indirect effect through v_D, as the default payoff is shown in Figure 7 to increase in λ. As for φ, accommodation proceeds along the debt-level margin alone: $PD_S^\star$ is unchanged.

Much the same phenomenon is at work in Figure 5c, albeit in the opposite direction: A larger loss of output in default increases the cost of default, thereby strengthening the restraint on the government and increasing maximum and optimal debt and borrowing. This direct effect is compounded by the indirect effect through v_D, as the default payoff is shown in Figure 7 to decrease in τ. Although $PD_S^\star$ is essentially unchanged, it initially very slightly decreases, thereby constituting one instance in which a change in restraint is accommodated along the two margins of debt level and default probability.

An increase in risk-aversion γ has barely an effect on the variables of interest (Figure 5d), as do increases in the interest rate r (Figure 6b) and in the mean growth rate μ (Figure 6c). The decrease of debt and borrowing in the importance of future payoffs θ (Figure 6a) suggests that, over the range of values considered at least, the indirect effect of θ through v_D dominates the direct effect: the restraint is weakened despite the greater importance attached future payoffs; debt, borrowing, and the probability of default decrease. Note that the sensitivity of $PD_S^\star$ to θ dwarfs that to all other parameters.

Figure 6d is interesting in that it is one for which the change in optimal debt and borrowing diverges from that of the probability of default. The essentially unchanged maximum feasible debt suggests that the restraint on the government is unchanged, perhaps because the strengthening direct effect and the weakening indirect effect cancel each other (higher volatility further endangers future payoffs through increased default; it increases the default payoff through a higher expected growth rate).[31] Absent a change in maximum debt, optimal debt decreases to account for the now larger possibility of default, but not so much as to prevent an increase in the probability of default.

9 Growth Collapses

We now add the possibility of growth collapses (Rietz, 1988; Barro, 2006; Barro and Ursua, 2011). Specifically, we assume

$$\log(g) = \mu + u - z;$$

u and v are mutually independent, $u \sim N(0, \sigma^2)$, and

$$v = \begin{cases} z & \text{with probability } p \\ 0 & \text{with probability } 1-p \end{cases}$$

where z is distributed as

$$f(z) = \begin{cases} \rho e^{-\rho(z-z_0)} & \text{if } z \geqslant z_0. \\ 0 & \text{otherwise.} \end{cases}$$

We set $p = 0.01$, $\rho = 4.5$, $z_0 = -\log(1 - 0.095)$; μ and σ remain as before. There is a 1% probability that there be a growth collapse; collapse, if it should occur, involves a decline in GDP of at least 9.5% and is exponentially distributed with rate 4.5.

The calibrated values are shown in the second row of Table 2. As expected, maximum and optimal debt and borrowing decrease whereas default probabilities increase. The government adjusts along the two available margins to the possibility of a growth collapse that would leave the government with insufficient resources to service its debt; the first margin will be recalled to be the debt level, the second the default probability. The probability of default at optimal debt is now $PD_E^\star = 0.953\%$, representing a one in a century frequency of default; optimal debt is $d_E^\star = 67.8\%$, MSD is $d_M = 71.5\%$.

Figure 8 shows the sensitivity of maximum and optimal debt and their corresponding proceeds and associated probabilities of default to the rate of the distribution ρ, the probability that there be a collapse in growth p, and the minimum decline in output such collapse involves z_0. Panel (a) shows that the rate of the distribution has practically no effect on debt and borrowing. There is a slight decrease in the probability of default at optimal debt, reflecting a very slight increase in the ability to forestall default even if there should be a collapse in growth when losses become more concentrated at their 9.5% lower bound. Panel (c) shows that increasing that lower bound from zero to 10% decreases debt and proceeds and increases the default probability; surprisingly perhaps, this does not extend beyond 10%: once the collapse in growth is such that

[31] Recall that $E[g] = \exp\left(\mu + \sigma^2/2\right)$ for g lognormally distributed.

default occurs, the ability repeatedly to raise new debt to servicing existing debt is lost, how large the collapse in growth is beyond this level does not matter. This suggests that it is the occurrence rather than the extent of default that matters.

This intuition is confirmed by Panel (b), which shows the variation of the values of interest in the probability p that collapse occurs. There is now a marked decrease in debt and borrowing, and a marked increase in default probabilities as collapse becomes more likely. Lenders react to the greater probability of collapse by requiring a higher implicit interest rate (the ratios d_M/b_M and $d_E^\star/b_E^\star$ increase from 1.024 to 1.044 and from 1.015 to 1.035, respectively, as p increases from 0.5% to 2.5%; see Figure 9). The government responds along the two margins of quantity and quality, issuing less and riskier debt.

10 The United States

We briefly compute for the United States the values we have computed for the Euro Area. We set the real interest rate r to match the return on 1-year Treasury bonds over the period 1955-2014 (series GS1 in FRED), net of the GDP deflator (GDPDEF in FRED); $r = 1.85\%$. We set the output process to match the mean and volatility of per-capital output (A939RX0Q048SBEA); $\mu = 1.94\%$ and $\sigma = 2.13\%$. The results are shown in Table 4. Those in the first row can be interpreted in light of the sensitivity analysis of Section 7.2, Figures 4a and 4b in particular: the US's larger mean growth rate μ increases maximum and optimal debt and proceeds; the US's larger interest rate r dampens such increase. The probabilities of default at MSD and at optimal debt are essentially unchanged, reflecting these probabilities' practically nil sensitivity to μ and r. The volatility of the growth rate being essentially the same in the Euro Area and the United States, it likely plays no role in the present comparison. The results in the second row, which pertain to the case of a collapse in growth, can by and large be interpreted in a manner similar to those in the first row.

US public debt stands at around 100% of GDP, 15% above $d_M = 85\%$ and 16% above $d_E^* = 84\%$. That the Fed holds about 15% of US public debt is a – welcome – coincidence. It was not part of the calibration of the model.

11 Conclusion

We have revisited the issue of optimal sovereign debt, assuming self-interested governments engaging in excusable default where most previous work had assumed benevolent governments engaging in strategic defaults. Our assumption of self-interested government is more in accordance with the Public Choice Theory of government than is the alternative assumption of benevolent government; our assumption of excusable default is in accordance with extensive empirical evidence that documents governments' extreme reluctance to default: governments are no doubt mindful of the loss of power that generally follows default.

Our calibrated optimal debt level (81% of GDP) is well above those obtained under the assumption of strategic default (ranging from 1% to 38%), and much closer to those observed in practice (often exceeding 100%). Lenders more readily lend to governments they expect to do their utmost to avoid default than to governments they fear continuously trade off the costs and benefits of default; governments exploit such readiness to reach optimal debt levels only very slightly below those of maximum sustainable debt (MSD). The very low probability of default at MSD, a consequence of the very low volatility of growth, keeps optimal debt close to MSD, which to a borrowing government represents the most advantageous trade-off between payment promised and proceeds received.

We have found optimal debt to be most sensitive to a country's maximum primary surplus, the mean and volatility of its growth rate, and the interest rate. Incorporating the possibility of growth collapses in our calibration, we have raised the probability of default at optimal debt from an implausibly low 0.106% to a much more realistic 0.953%, corresponding to a one in a century frequency of default for an advanced economy such as the US. The now lower optimal debt at 67.8% and MSD at 71.5%, much below prevailing debt levels, suggest the need to incorporate governments' ability to direct central banks' purchases of government debt into our analysis. We leave such extension to future work.

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A Tables

Table 1: Parameter Values

	r (%)	μ (%)	σ (%)	α	φ	θ	λ	τ	γ	β
Excusable Default	1.04	1.02	2.12	0.05	0.50	0.60	NA	NA	0.50	0.95
Strategic Default	1.04	1.02	2.12	NA	1.00	1.00	0.734	0.02	0.50	0.95

Note: NA stands for Not Applicable. Data on output growth are obtained from the Area Wide Model database, Data for the interest rate are obtained from the International Finance Statistics database from the International Monetary Fund, and correspond to the Government bond yields for the euro area. The sample ranges from 1990 to 2013.

Table 2: Excusable Default

	φ	θ	d_M (%)	$d^\star_E$ (%)	b_M (%)	$b^\star_E$ (%)	PD_M (%)	$PD^\star_E$ (%)
Excusable	0.50	0.60	83.222	82.083	81.733	81.151	0.765	0.106
Excusable	1.00	0.60	83.222	81.815	81.733	80.921	0.765	0.062
Excusable	0.50	1.00	83.222	79.924	81.733	79.099	0.765	0.001
Excusable	1.00	1.00	83.222	79.679	81.733	78.857	0.765	0.000
Collapse	0.50	0.60	71.533	67.822	69.551	66.483	1.757	0.953

Note: The subscript M denotes maximum values, the superscript $\star$ denotes optimal values. The Growth Collapse model assumes that the growth process takes the form $g = \mu + u - z$, where $z > 0$ with probability p and 0 with probability $1 - p$. When positive, z has pdf $f(z) = \rho \exp(-\rho(z - z_0))$ if $z > z_0$, 0 otherwise. In the application, we set $p = 0.01$, $\rho = 4.5$ and $z_0 = -\log(1 - 0.095)$ implying that only GDP drops of more than 9.5% are considered collapses (see Barro (2006) and Barro and Ursua (2014)).

Table 3: Strategic Default

φ	θ	ω_S (%)	$d^\star_S$ (%)	$b^\star_S$ (%)	$PD^\star_S$ (%)	v_D
1.00	1.00	2.876	2.698	2.669	0.026	44.343
0.50	1.00	1.443	1.353	1.339	0.026	31.356
1.00	0.60	4.539	4.321	4.263	0.296	4.680
0.50	0.60	2.275	2.162	2.133	0.296	3.310

Table 4: Excusable Default (US Calibration)

	φ	θ	d_M (%)	$d_E^\star$ (%)	b_M (%)	$b_E^\star$ (%)	PD_M (%)	$PD_E^\star$ (%)
Excusable	0.50	0.60	85.534	84.360	83.336	82.740	0.768	0.106
Collapse	0.50	0.60	73.481	70.328	70.879	68.379	1.757	0.973

Note: In this calibration, $r = 1.85\%$, $\mu = 1.94\%$ and $\sigma = 2.13\%$. The subscript M denotes maximum values, the superscript $\star$ denotes optimal values. The Growth Collapse model assumes that the growth process takes the form $g = \mu + u - z$, where $z > 0$ with probability p and 0 with probability $1 - p$. When positive, z has pdf $f(z) = \rho \exp(-\rho(z - z_0))$ if $z > z_0$, 0 otherwise. In the application, we set $p = 0.01$, $\rho = 4.5$ and $z_0 = -\log(1 - 0.095)$ implying that only GDP drops of more than 9.5% are considered collapses (see Barro (2006) and Barro and Ursua (2014)).

B Figures

Figure 1: Probability of Default PD

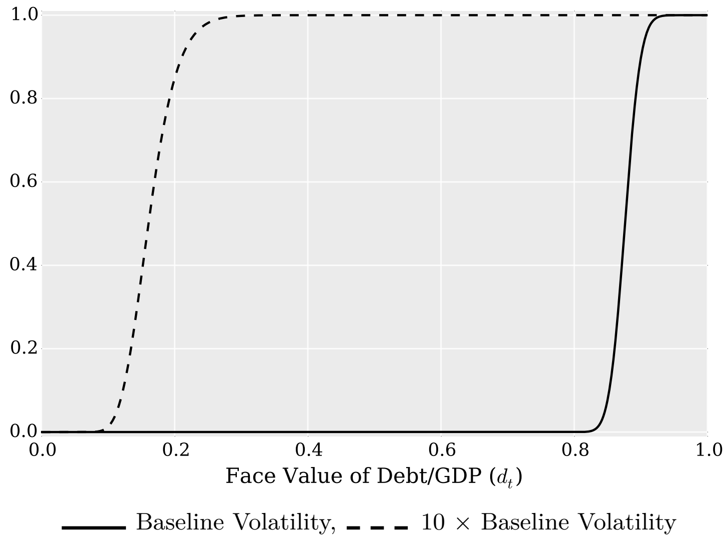


Figure 2: Borrowing proceeds b

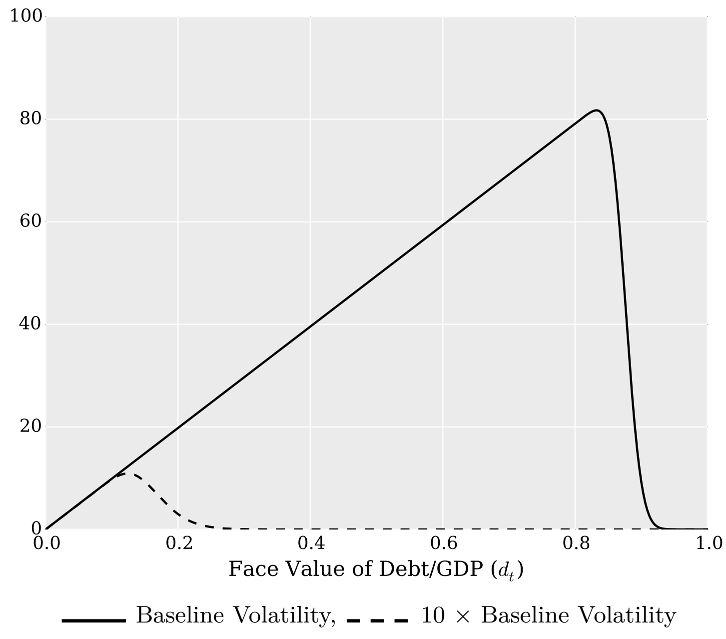
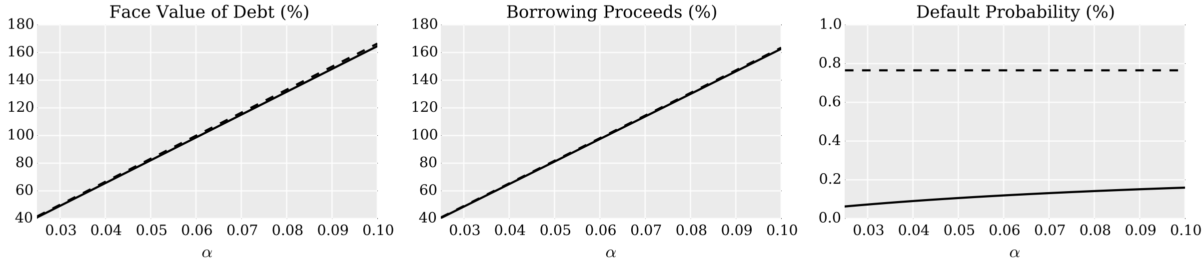
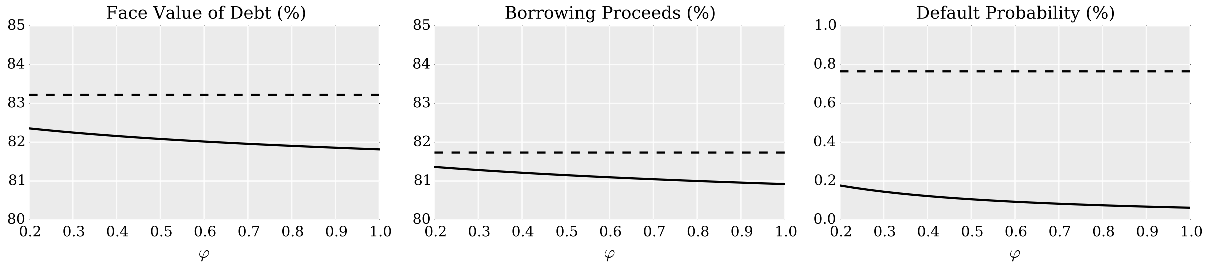


Figure 3: Sensitivity Analysis: Excusable Default Model (I)

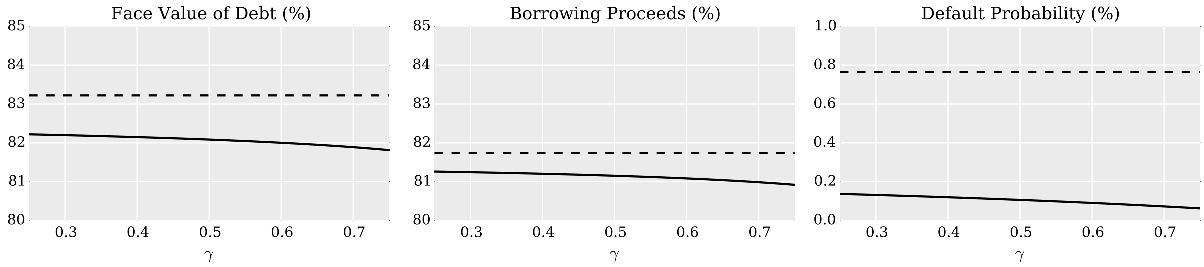
(a) Variation in α



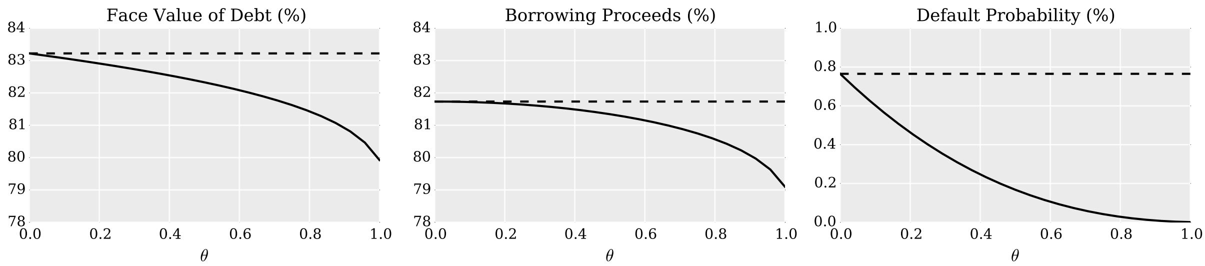
(b) Variation in φ



(c) Variation in γ



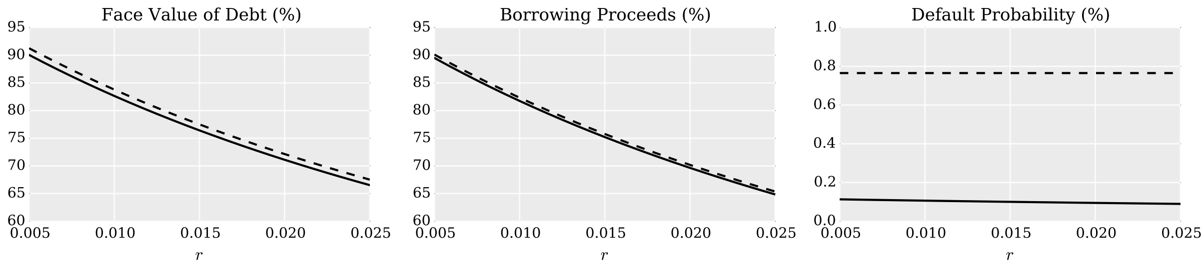
(d) Variation in θ



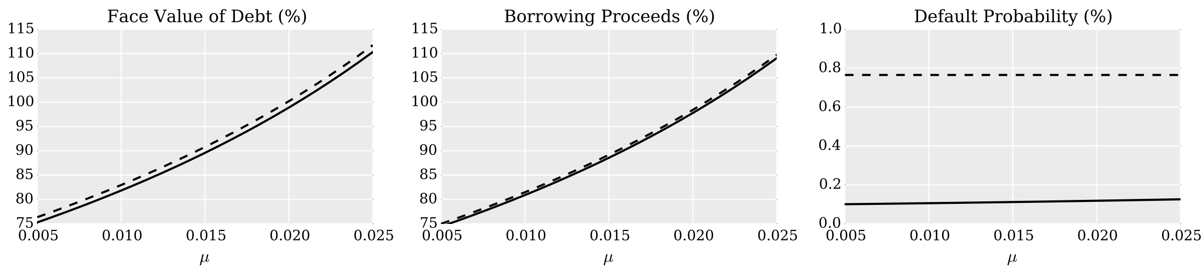
Maximum, Optimal

Figure 4: Sensitivity Analysis: Excusable Default Model (II)

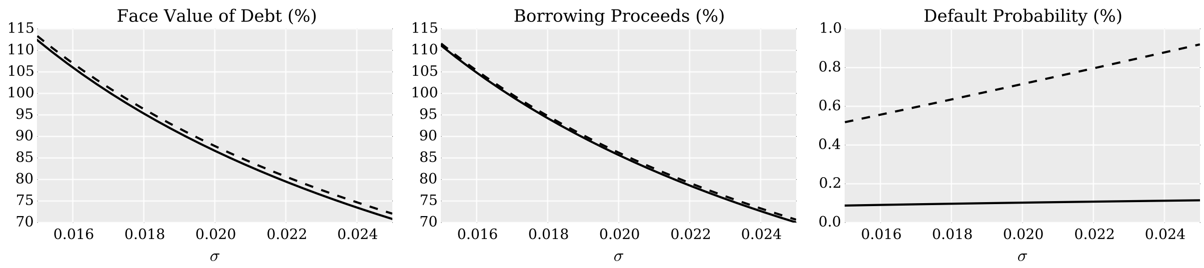
(a) Variation in r



(b) Variation in μ



(c) Variation in σ



(d) Variation in β

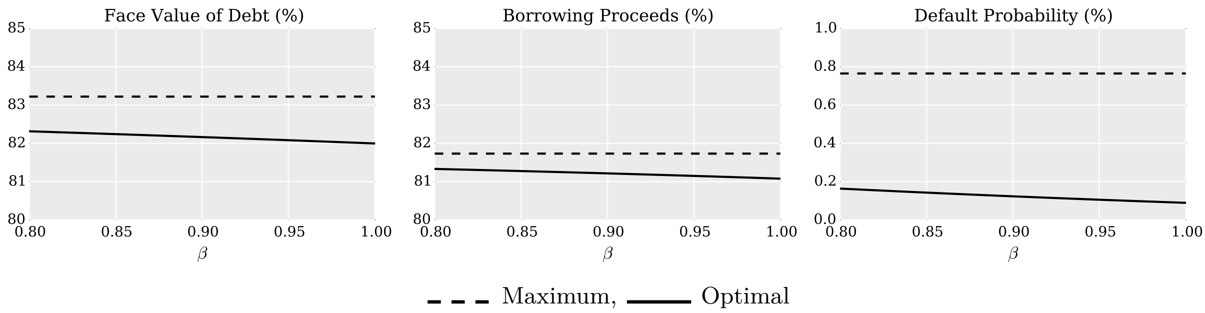
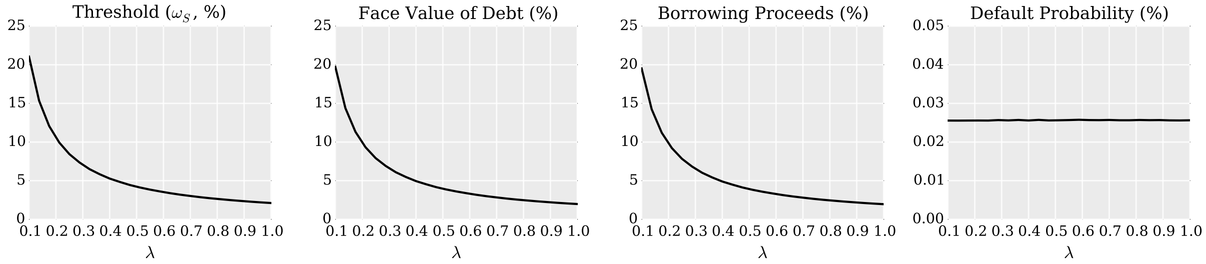


Figure 5: Sensitivity Analysis: Strategic Default Model (I)

(a) Variation in φ

(b) Variation in λ



(c) Variation in τ

(d) Variation in γ

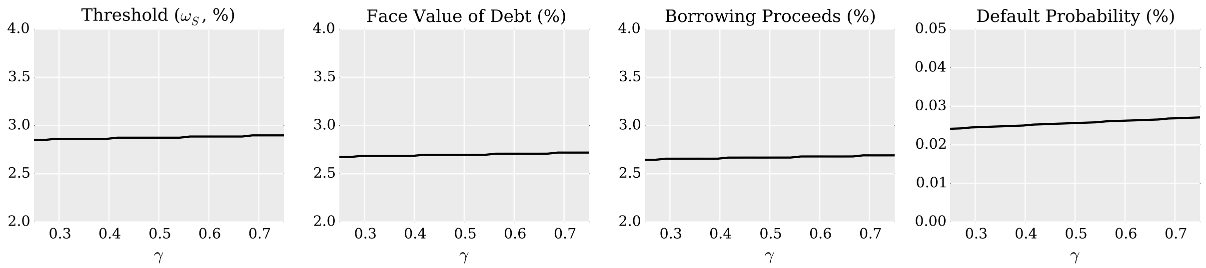
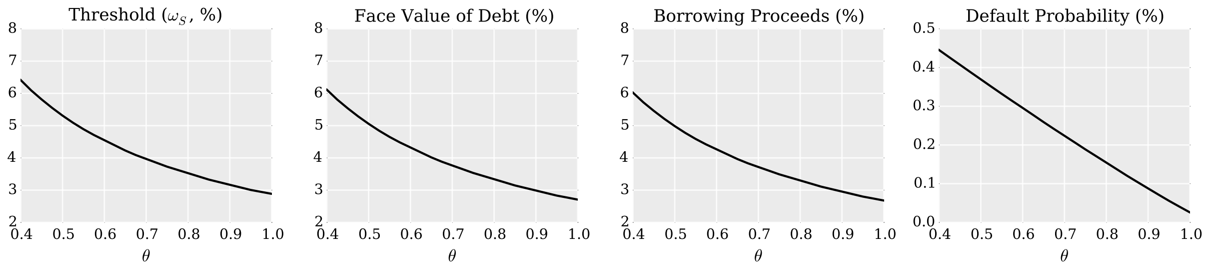
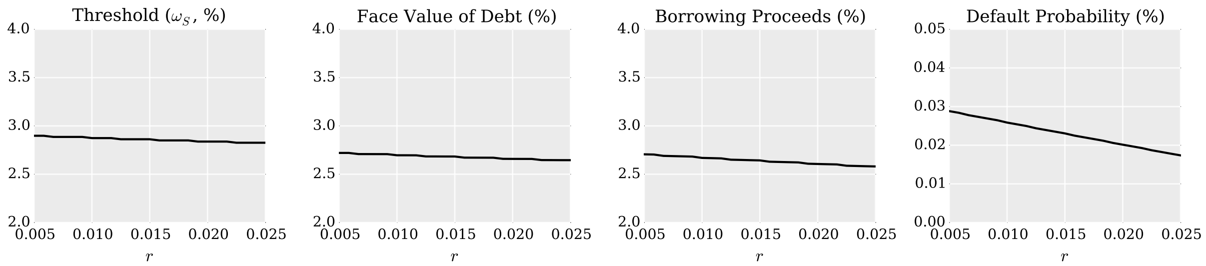


Figure 6: Sensitivity Analysis: Strategic Default Model (II)

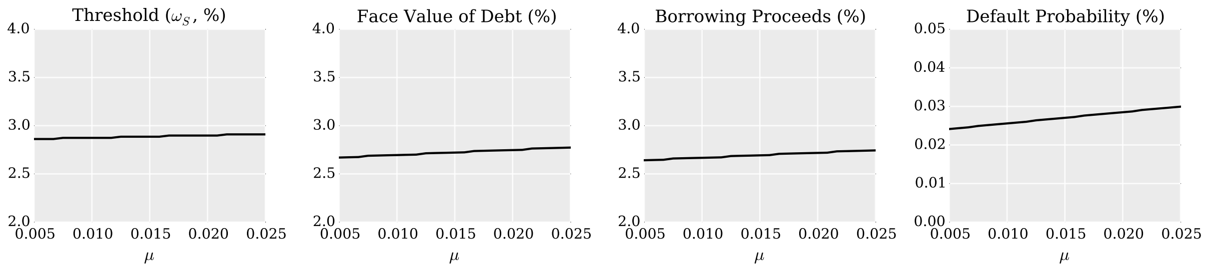
(a) Variation in θ



(b) Variation in r



(c) Variation in μ



(d) Variation in σ

Figure 7: Equilibrium Default Payoff v_D

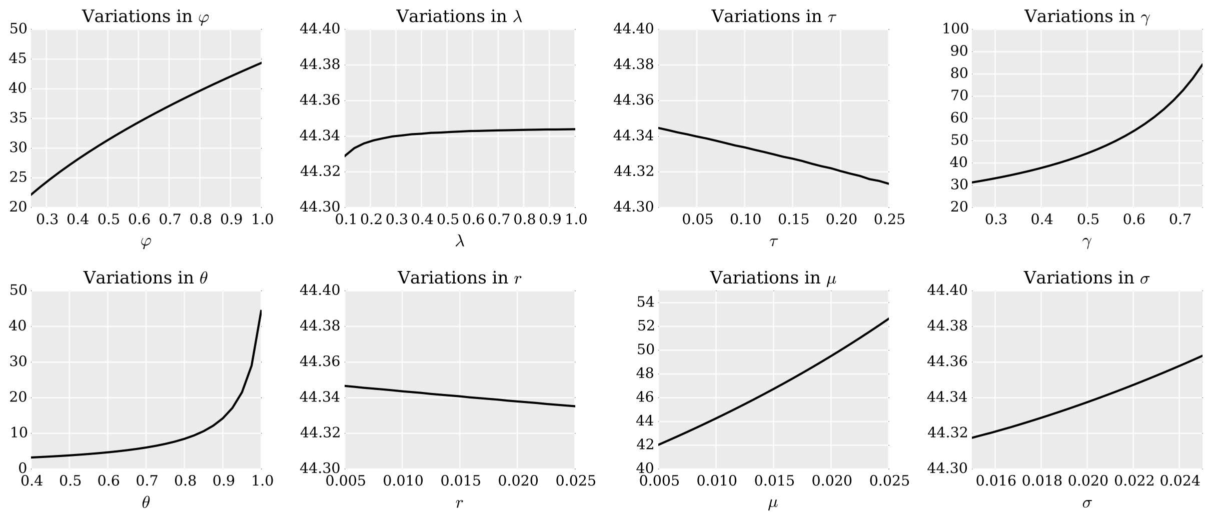
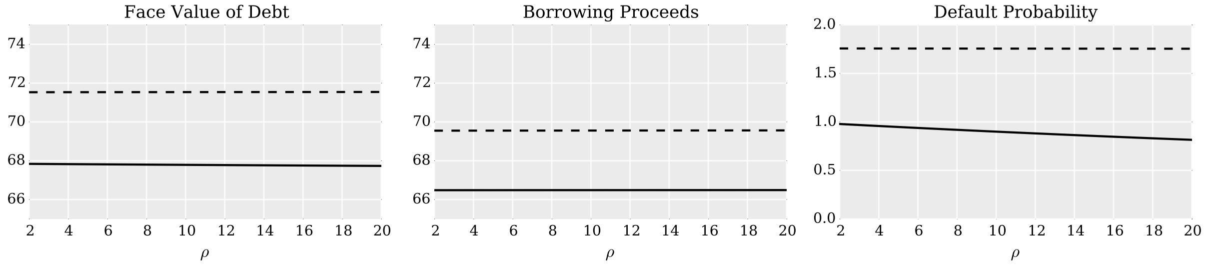
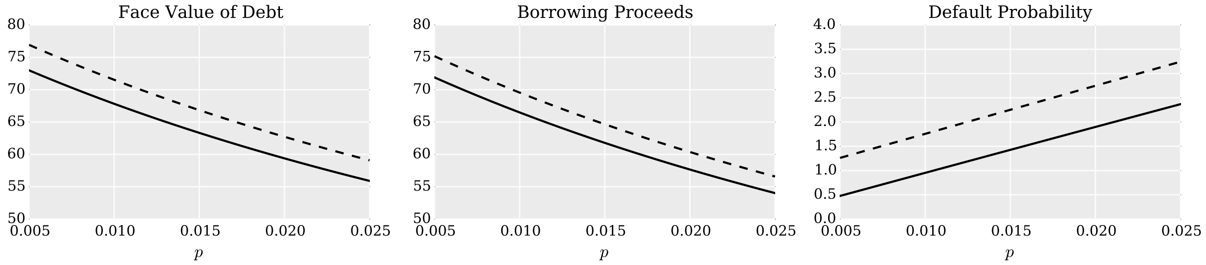


Figure 8: Sensitivity Analysis: Excusable Default Model with Disasters

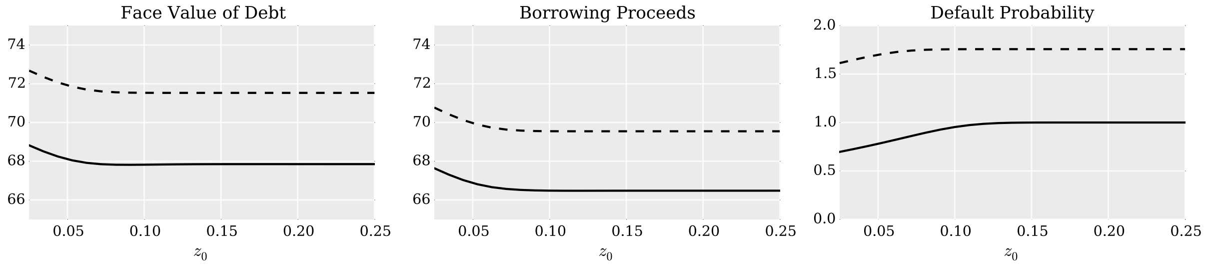
(a) Variation in ρ



(b) Variation in p

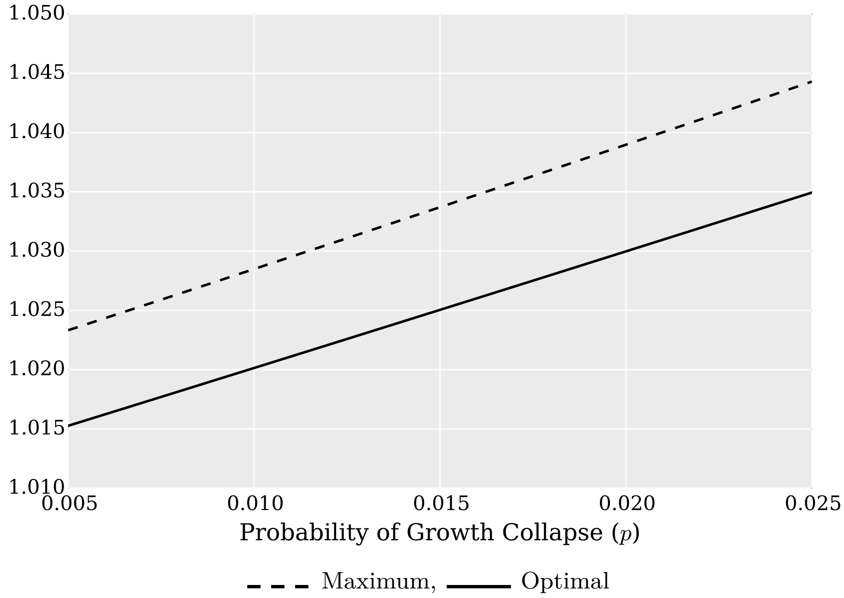


(c) Variation in z_0



Maximum, Optimal

Figure 9: Excusable Default Model with Growth Collapses, Debt/Proceeds Ratio



C Proofs

Proof of Proposition 1 Define the operator T such that

$$T(v_E)(\omega) = \max_{g_E} u\left(\varphi + \frac{\alpha + b_M}{1+r} g_E(1 - F(g_E)) - \omega\right) + \beta\theta \int_{g_E}^{\infty} v_E\left((\alpha + b_M)\frac{g_E}{g}\right) g^{1-\gamma} dF(g)$$

Proving that the Bellman equation satisfies the Blackwell condition involves proving that it satisfies both the monotonicity and the discounting properties.

Monotonicity: Consider two candidate solutions v and w, such that $v(\omega) \leqslant w(\omega)$; denote $g_V^\star$ and $g_W^\star$ their corresponding maximizers. We have

$$\begin{aligned}
T(v)(\omega) &= u\left(\varphi + \frac{\alpha + b_M}{1+r} g_V^\star(1 - F(g_V^\star)) - \omega\right) + \beta\theta \int_{g_V^\star}^{\infty} v\left((\alpha + b_M)\frac{g_V^\star}{g}\right) g^{1-\gamma} dF(g) \\
&\leqslant u\left(\varphi + \frac{\alpha + b_M}{1+r} g_V^\star(1 - F(g_V^\star)) - \omega\right) + \beta\theta \int_{g_V^\star}^{\infty} w\left((\alpha + b_M)\frac{g_V^\star}{g}\right) g^{1-\gamma} dF(g) \\
&\leqslant u\left(\varphi + \frac{\alpha + b_M}{1+r} g_W^\star(1 - F(g_W^\star)) - \omega\right) + \beta\theta \int_{g_W^\star}^{\infty} w\left((\alpha + b_M)\frac{g_W^\star}{g}\right) g^{1-\gamma} dF(g) \\
&= T(w)(\omega),
\end{aligned}$$

which establishes monotonicity.

Discounting: Let a be a strictly positive constant. We have

$$\begin{aligned}
T(v+a)(\omega) =& \max_{g_E} u\left(\varphi + \frac{\alpha + b_M}{1+r} g_E(1 - F(g_E)) - \omega\right) + \beta\theta \int_{g_E}^{\infty} \left(v_E\left((\alpha + b_M)\frac{g_E}{g}\right) + a\right) g^{1-\gamma} dF(g) \\
&\leqslant \max_{g_E} u\left(\varphi + \frac{\alpha + b_M}{1+r} g_E(1 - F(g_E)) - \omega\right) + \beta\theta \int_{g_E}^{\infty} v_E\left((\alpha + b_M)\frac{g_E}{g}\right) g^{1-\gamma} dF(g) \\
&+ a \max_{g_E} \beta\theta \int_{g_E}^{\infty} g^{1-\gamma} dF(g) \\
&= T(v)(\omega) + a \max_{g_E} \beta\theta \int_{g_E}^{\infty} g^{1-\gamma} dF(g).
\end{aligned}$$

As $\frac{d}{dg_E} \int_{g_E}^{\infty} g^{1-\gamma} dF(g) < 0$, we have

$$\max_{g_E} \int_{g_E}^{\infty} g^{1-\gamma} dF(g) < \int_0^{\infty} g^{1-\gamma} dF(g) = E(g^{1-\gamma}).$$

A sufficient condition for discounting is therefore that $\beta\theta E(g^{1-\gamma}) < 1$. ■

D Solution Method

D.1 Excusable Default Model

Solving the excusable default model amounts to solving the Bellman equation

$$v_E(\omega_t) = \max_{g_{E,t+1}} u\left(\varphi + \frac{\alpha + b_M}{1+r} g_{E,t+1}\left[1 - F(g_{E,t+1})\right] - \omega_t\right) + \theta\beta \int_{g_{E,t+1}}^{\infty} v_E\left((\alpha + b_M)\frac{g_{E,t+1}}{g}\right) g^{1-\gamma} dF(g).$$

Using the assumption that growth is log-normally distributed, we can rewrite the Bellman equation as

$$v_E(\omega_t) = \max_{x_{E,t+1}} \frac{\left(\varphi + \frac{\alpha+b_M}{1+r}\exp(\mu + \sigma x_{E,t+1})(1 - \Phi(x_{E,t+1})) - \omega_t\right)^{1-\gamma}}{1-\gamma} + \theta\beta \int_{x_{E,t+1}}^{\infty} v_E\left((\alpha + b_M)\exp(\sigma(x_{E,t+1} - x))\right)\exp((1-\gamma)(\mu + \sigma x)) d\Phi(x),$$

where $x \equiv (\log(g) - \mu)/\sigma$ and $x_{E,t+1} \equiv (\log(g_{E,t+1}) - \mu)/\sigma$. The preceding involves the computation of the integral

$$\begin{aligned} I(x_E) &= \int_{x_E}^{\infty} v_E\left((\alpha + b_M)\exp(\sigma(x_E - x))\right)\exp((1-\gamma)(\mu + \sigma x)) d\Phi(x) \\ &= \frac{\exp((1-\gamma)\mu)}{\sqrt{2\pi}} \int_{x_E}^{\infty} \exp((1-\gamma)\sigma x) v_E\left((\alpha + b_M)\exp(\sigma(x_E - x))\right)\exp(-x^2/2) dx \\ &= \frac{\exp((1-\gamma)(\mu + \sigma x_E))}{\sqrt{2\pi}} \int_0^{\infty} \exp((1-\gamma)\sigma s) v_E\left((\alpha + b_M)\exp(-\sigma s)\right)\exp(-(s + x_E)^2/2) ds, \end{aligned}$$

where $s \equiv x - x_E$. Multiplying and dividing by $\exp(-s)$, we obtain

$$I_2(x_E) = \frac{\exp((1-\gamma)(\mu + \sigma x_E))}{\sqrt{2\pi}} \int_0^{\infty} \psi(s; x_E)\exp(-s) ds,$$

where

$$\psi(s; x_E) = \exp((1 + (1-\gamma)\sigma)s - (s + x_E)^2/2) v\left((\alpha + b_M)\exp(-\sigma s)\right),$$

which is evaluated using a Gauss-Laguerre quadrature method.

The algorithm is then as follows

1. Set a grid of values for ω: $\Omega = \{\omega_1, \ldots, \omega_n\}$, and a grid for x_E, $\Xi = \{x_{E1}, \ldots, x_{Em}\}$. Use 1,000 grid points for Ω and 20,000 grid points for Ξ.

2. Conjecture a value function $v_{E,i}(\omega)$, $i = 0$.

3. Given the value function conjectured at iteration i, evaluate

$$\Psi(\omega, x_E) = \frac{\left(\varphi + \frac{\alpha+b_M}{1+r}\exp(\mu + \sigma x_{E,t+1})(1 - \Phi(x_{E,t+1})) - \omega_t\right)^{1-\gamma}}{1-\gamma} + \theta\beta I(x_E),$$

   for each pair $(\omega, x_E) \in (\Omega \times \Xi)$, where integrals are evaluated using a 100 nodes Gauss-Laguerre quadrature method.

4. Find $\Psi^\star(\omega) = \max_{x_E} \Psi(\omega, x_E)$ and the associated $x_E^\star(\omega) = \text{argmax}_{x_E} \Psi(\omega, x_E)$ on the grid and update the value function

$$v_{E,i+1}(\omega) = T(v_{E,i})(\omega) = \Psi^\star(\omega).$$

5. If $\mathcal{E}_E = \|v_{E,i+1}(\omega) - v_{E,i}(\omega)\|_\infty < \varepsilon$, $\varepsilon > 0$, then stop, else go back to 3.

After having achieved convergence, compute

$$\begin{aligned} g_E^{\star}(\omega) &= \exp(\mu + \sigma x_E^{\star}(\omega)), \\ d_E^{\star}(\omega) &= (\alpha + b_M)\exp(\mu + \sigma x_E^{\star}(\omega)), \\ b_E^{\star}(\omega) &= \frac{\alpha + b_M}{1+r}\exp(\mu + \sigma x_E^{\star}(\omega))\left[1 - \Phi\left(x_E^{\star}(\omega)\right)\right], \\ PD_E^{\star}(\omega) &= \Phi\left(x_E^{\star}(\omega)\right). \end{aligned}$$

D.2 Strategic Default model

Solving the strategic default model amounts to solving the system of equations

$$\begin{aligned} v_S(\omega_t) = \max\Bigg\{ v_D, \max_{g_{S,t+1}} & \frac{\left(\varphi + \frac{\omega_S}{1+r} g_{S,t+1}(1 - F(g_{S,t+1})) - \omega_t\right)^{1-\gamma}}{1-\gamma} \\ & + \beta\theta\left[\int_0^{g_{S,t+1}} v_D g^{1-\gamma} dF(g) + \int_{g_{S,t+1}}^{\infty} v_S\left(\omega_S \frac{g_{S,t+1}}{g}\right) g^{1-\gamma} dF(g)\right]\Bigg\}, \\ v_D &= \frac{(\varphi(1-\tau))^{1-\gamma}}{1-\gamma} + \beta\theta\int_0^{\infty} \lambda v_S(0) g^{1-\gamma} + (1-\lambda) v_D g^{1-\gamma} dF(g), \\ v_S(\omega_S) &= v_D. \end{aligned}$$

Using the assumption that growth is log-normally distributed, we can rewrite the system of equation as

$$\begin{aligned} v_S(\omega_t) = \max\Bigg\{ v_D, \max_{x_{S,t+1}} & \frac{\left(\varphi + \frac{\omega_S}{1+r}\exp(\mu + \sigma x_{S,t+1})(1 - \Phi(x_{S,t+1})) - \omega_t\right)^{1-\gamma}}{1-\gamma} \\ & + \beta\theta\Bigg[v_D \int_{-\infty}^{x_{S,t+1}} \exp((1-\gamma)(\mu + \sigma x)) d\Phi(x) \\ & + \int_{x_{S,t+1}}^{\infty} v_S\left(\omega_S \exp(\sigma(x_{S,t+1} - x))\right)\exp((1-\gamma)(\mu + \sigma x)) d\Phi(x)\Bigg]\Bigg\} \\ v_D &= \frac{(\varphi(1-\tau))^{1-\gamma}}{1-\gamma} + \beta\theta(\lambda v_S(0) + (1-\lambda) v_D)\exp\left((1-\gamma)\mu + (1-\gamma)^2\frac{\sigma^2}{2}\right) \\ v_S(\omega_S) &= v_D, \end{aligned}$$

where $x \equiv (\log(g) - \mu)/\sigma$ and $x_{S,t+1} \equiv (\log(g_{S,t+1}) - \mu)/\sigma$. The preceding involves the computation of the two integrals

$$\begin{aligned} I_1(x_S) &= \int_{-\infty}^{x_S} \exp((1-\gamma)(\mu + \sigma x)) d\Phi(x), \\ I_2(x_S) &= \int_{x_S}^{\infty} v_S\left(\omega_S \exp(\sigma(x_S - x))\right)\exp((1-\gamma)(\mu + \sigma x)) d\Phi(x). \end{aligned}$$

Using the explicit form of the normal distribution, it is straightforward to compute the first integral

$$I_1(x_S) = \exp\left((1-\gamma)\mu + (1-\gamma)^2\frac{\sigma^2}{2}\right)\Phi\left(x - (1-\gamma)\sigma\right).$$

Now turn to the second integral. We have

$$\begin{aligned} I_2(x_S) &= \int_{x_S}^{\infty} v_S\left(\omega_S \exp(\sigma(x_S - x))\right)\exp((1-\gamma)(\mu + \sigma x)) d\Phi(x) \\ &= \frac{\exp((1-\gamma)\mu)}{\sqrt{2\pi}}\int_{x_S}^{\infty} \exp((1-\gamma)\sigma x) v_S\left(\omega_S \exp(\sigma(x_S - x))\right)\exp(-x^2/2) dx \\ &= \frac{\exp((1-\gamma)(\mu + \sigma x_S))}{\sqrt{2\pi}}\int_0^{\infty} \exp((1-\gamma)\sigma s) v_S\left(\omega_S \exp(-\sigma s)\right)\exp(-(s + x_S)^2/2) ds, \end{aligned}$$

where $s \equiv x - x_S$. Multiplying and dividing by $\exp(-s)$, we obtain

$$I_2(x_S) = \frac{\exp((1-\gamma)(\mu + \sigma x_S))}{\sqrt{2\pi}}\int_0^{\infty} \psi(s; x_S)\exp(-s) ds,$$

where

$$\psi(s;x_S) = \exp((1+(1-\gamma)\sigma)s - (s+x_S)^2/2)v\left(\omega_S \exp(-\sigma s)\right),$$

which is evaluated using a Gauss-Laguerre quadrature method.

The algorithm is then as follows

1. Set a grid of values for ω: $\Omega = \{\omega_1, \dots, \omega_n\}$, and a grid for x_S, $\Xi = \{x_{S1}, \dots, x_{Sm}\}$. We use 500 grid points for Ω and 20,000 grid points for Ξ.
2. Conjecture a value function $v_{S,i}(\omega)$, $i = 0$, and an initial value for $v_{D,i}$, $i = 0$. We use
$$v_{D,0} = \frac{\frac{(\varphi(1-\tau))^{1-\gamma}}{1-\gamma} + \beta\theta\lambda v_{S,0}(0)\mathbb{E}[g^{1-\gamma}]}{1-\beta\theta(1-\lambda)\mathbb{E}[g^{1-\gamma}]}$$
and an initial threshold value $\omega_{S,i}$.
3. Given values conjectured at iteration i, evaluate :
$$\Psi(\omega, x_S) = \frac{\left(\varphi + \frac{\omega_S}{1+r}\exp(\mu+\sigma x_S)(1-\Phi(x_S)) - \omega\right)^{1-\gamma}}{1-\gamma} + \beta\theta(I_1(x_S)v_{D,i} + I_2(x_S)),$$
for each pair $(\omega, x_S) \in (\Omega \times \Xi)$, where integrals are evaluated using a 100 nodes Gauss-Laguerre quadrature method.
4. Find $\Psi^\star(\omega) = \max_{x_S} \Psi(\omega, x_S)$ and the associated $x_S^\star(\omega) = \operatorname{argmax} \Psi(\omega, x_S)$ on the grid and update the value functions
$$\begin{aligned} v_{S,i+1}(\omega) &= T\left(v_{S,i}\right)(\omega) = \max(v_{D,i}, \Psi^\star(\omega)), \\ v_{D,i+1} &= T\left(v_{D,i}\right) = \frac{(\varphi(1-\tau))^{1-\gamma}}{1-\gamma} + \beta\theta\left[\lambda v_{S,i}(0) + (1-\lambda)v_{D,i}\right]\exp\left((1-\gamma)\mu + (1-\gamma)^2\frac{\sigma^2}{2}\right). \end{aligned}$$
5. Update the threshold value $\omega_{S,i+1}$ that solves
$$v_{S,i+1}(\omega_{S,i+1}) = v_{D,i+1}.$$
6. Compute $\mathcal{E}_S = \|v_{S,i+1}(\omega) - v_{S,i}(\omega)\|_\infty$, $\mathcal{E}_A = \|v_{A,i+1} - v_{A,i}\|_\infty$ and $\mathcal{E}_\omega = \|\omega_{S,i+1} - \omega_{S,i}\|_\infty$. If $\max(\mathcal{E}_S, \mathcal{E}_A, \mathcal{E}_\omega) < \varepsilon$, $\varepsilon > 0$, then stop, else go back to 3.

After having achieved convergence, compute

$$\begin{aligned} g_S^\star(\omega) &= \exp(\mu + \sigma x_S^\star(\omega)), \\ d_S^\star(\omega) &= \omega_S \exp(\mu + \sigma x_S^\star(\omega)), \\ b_S^\star(\omega) &= \frac{\omega_S}{1+r}\exp(\mu + \sigma x_S^\star(\omega))\left[1 - \Phi\left(x_S^\star(\omega)\right)\right], \\ PD_S^\star(\omega) &= \Phi\left(x_S^\star(\omega)\right). \end{aligned}$$