The Axiomatic Approach to Selection from Sets*

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Abstract

We study the problem of evaluating whether the selection from a set is close to the ranking of the set determined by a measurable criterion. Our main result is that three axioms, two naturally capturing "dominance", and a stronger one imposing a form of symmetry in the comparison of selections are sufficient to order completely all the selections from sets, according to how close they are to the ranking. This is given by a very simple index, which is a linear function of the sum of the ranks of the selected elements. The paper ends by relating this index to the existing literature on distance between rankings, and also offers a practical application of the index.

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1 Introduction

Selections are typically made according to a varying blend of objective measures and subjective judgments. A sport coach might base her team choice on the recent performance of individuals in her squad (batting and bowling/pitching averages, tennis rankings, trial results for track and field, and so on), and on her sense of who is the best person for each given role, given the expected conditions. Many universities do not strictly follow SATs results and school exam grades when choosing whom to admit, but take into account a student's social background and his potential contribution to desirable characteristics of the student body, like diversity. Applicants for academic jobs might be ranked according to bibliometric measures, but the appointment panel's judgment often leads to decision that do not map precisely into the ranking.[1] Large and complex procurement contracts often demand the subtle evaluation of complex qualitative elements, and lowest price is seldom the only criterion used to award these contracts.[2] And so forth and so on.

Observers and decision makers might be interested in some means of comparing the choices of different selectors. If, for want of a better term, the property of following the measurable dimension is labelled "rankiness", someone might want to compare different selections, and determine which is more "rank-based".[3]

We can think of at least three conceptually distinct situations where this

[1]An example from recent implementation of policy which some readers will be familiar with is the extent by which bibliometric criteria should be used in the evaluation of university research departments. Unlike in Italy, the UK funding body was persuaded to allow panels not to adhere strictly to bibliometric measures of departmental output, but allow the latitude afforded by peer review. We reprise this theme in Section 5.

[2]As, for instance, the 1991 auctions for the 16 regional television franchises in the UK, when only half the franchises were awarded to the highest bidder; see Cabizza and De Fraja (1998), especially Table 1, pp 11-12.

[3]This term is chosen in analogy to its use in games of status (eg Hopkins and Kornienko 2009), or in the analysis of rank-dependent expected utility (Abdellaoui 2002).

comparison is meaningful. First, we may want to compare different selections from the same set. For example, a hiring committee chair may want to know how close each panel member's suggestion is to a bibliometric ranking of the applicants. Similarly for the judges of a book or film prize: critics may want to know how close their choices are to a market-determined (sales, box office earnings) ranking. Second, the comparison might be of selections from altogether different sets. A cricket fan may want to know whether Australia's selection for the Ashes team is more rank-based than England's. Or whether it is more rank-based than it was sixty years ago.[4] A basketball journalist may want to contend that a certain team's selection is more based on height than another's. A university whose admission policies are under scrutiny in court may want to argue that its admission policy is as based on SATs as those of comparable institutions. Or a government minister concerned about corruption in procurement contracts or personnel hiring may want to compare the rankiness of various commissioning boards or hiring panels, to identify and perhaps investigate atypical behaviour. In extreme, artificial cases, even totally unrelated sets may be compared: one might be interested in whether selection in academia is more rank-based than in sports. Finally, on a wider scale, rankiness is a helpful yardstick when assessing inequality of opportunity: it can serve as a measure of nepotism, telling how close selection into society's elite is to the ranking determined by family history, or of plutocracy, measured for example by the closeness of membership of Parliament to a person's position in the income distribution. The third kind

[4] Rankiness is at the basis of the popular book and film *Moneyball* (Lewis 2004), which tell the story of an obsessively rank-based baseball team with relatively scarce financial resources which was able systematically to outperform its much wealthier rivals; Hakes and Sauer (2006) confirm econometrically the book's intuition. Similarly, "artists and repertoire" talent spotters are being replaced in the music industry by detailed analyses of big data harvested from social media (Mukerji 2015). The nature of the sport makes the method less applicable to soccer (Anderson and Sally 2013).

of potential comparisons is for situations where both the selection and the selectors are the same, but there are two or more metrics in a set. In these cases, which are close to the topic of some related literature considered below (Kemeny 1959, Klamler 2008), the focus is on their relative importance in the selection. For example, a rugby analyst may want to know whether weight or speed is more important to be selected as a three-quarter for the Springboks. Or an external funder may want to know whether teaching or research are more important for promotion in a given university.

Comparing selections is straightforward only in the starkest cases. Sure, the selection of the best ranked is unquestionably more rank-based than one that selects a different element from the pool. But is a university which, from its 100 applicants, admits as students the second, eighth, ninth, twenty-second, and thirtieth ranked more rank-based than one that chooses the first five, and those ranked between 77 and 81 out of 200 applicants? Or in an even simpler example, is picking the second ranked out of ten candidates for a job more rank-based than selecting the third ranked out of twenty? Selection is of course a broader concept than full ranking: choosing the all-time best 20 in a list of 1000 footballers is different and less demanding than ranking the best 20. The relation between selection and ranking is explored in Section 4.

This paper proposes an axiomatic approach to comparing selections. We begin by requiring that rankiness satisfies three axioms: these, presented in the next section, are based on the idea of dominance in the comparison of sets (Barberà et al 2004, BPP in what follows). However, we depart at the outset from the literature reviewed in BPP which orders different subsets of a *given set*, in that we explicitly aim to compare selections from, namely subsets of, *different sets*.

Each of the axioms we propose takes a binary comparison between selections where the answer is unambiguous to the question as to which of the two selections is more rank-based, and requires that their relative rankiness be so determined. These three axioms are not characterising: there are different orderings of selections from sets which satisfies them. The paper continues by introducing the index of rankiness. In close analogy to the representation of a consumer's preference by a utility function, this attaches a real number to each selection, with higher numbers attached to more rank-based selections.

Our main result is Theorem 1. It shows that if one of the axioms is replaced by a stronger version, labelled "mirror invariance", then the index of rankiness is uniquely given by a simple linear function of the sum of the ranks of the selected elements. The mirror invariance axiom intuitively requires that if a change makes a selection more rank-based, then the mirror image of the selection – that is the selection of only the non-selected elements – is made less rank-based by the mirror image of the change. Thus, if one is willing to accept dominance, as defined in Axioms 2 and 3, and mirror invariance, stated in Axiom 4, then Theorem 1 establishes that selections from sets are completely ordered according to their rankiness by a simple function of the number of elements in the set, the number of selected elements, and the sum of the ranks of the selected elements.

A different viewpoint in the nature of the characterising axioms is obtained if the mirror invariance axiom is replaced by an equivalent one, labelled "position irrelevance", which intuitively requires that deviations from the selection which would be determined by the ranking have the same effect on the rankiness of the selection if they occur among the best or the worst ranked elements.

The paper continues by showing that our index of rankiness has a simple relation with the concept of distance between rankings originally proposed by Kendall (1938), and given an axiomatic foundation by Kemeny (1959). Specifically, Proposition 4 shows that if a selection is more rank-based than another selection from the same set, then the ranking of the set naturally induced by the first selection is nearer, in the sense of the Kendall-Tau distance, to the ranking of the set.

The paper is organised as follows: the dominance axioms and some preliminary results are presented in Section 2. The core of the paper is Section 3, which presents the index of rankiness, and strengthens one of the axioms to obtain a complete ordering of all selected sets. The relation with existing literature is explored in Section 4, and the paper ends with Section 5, which shows how the index can be used to assess the evaluation mechanism for promotion to professorship in Italian universities, and a brief conclusion.

2 Axioms of "rankiness"

We consider the framework in BBP, Section 3 (pp 902 ff). Let $\mathcal{N}$ be a completely ranked finite set. By this, we mean that $\mathcal{N}$ is a set with $N \in \mathbb{N}\setminus\{1\}$ elements, and on $\mathcal{N}$ a binary relation R is defined, which is transitive, complete and asymmetric. Thus, given any different three of its elements x, y, and z, then xRy and yRz imply xRz, and either xRy or yRx but not both. We therefore rule out ties between elements, consideration of which is important for practical purposes, but not straightforward.[5] We remark briefly in Section 4 below on possible approaches when some elements of the set are ranked equally, otherwise we restrict our attention to asymmetric

[5]While many rankings, for example, running times, or jump lengths, are continuous and the ties recorded in practice are generated by limitations of measurements, other measures, such as the number of goals scored, of wickets taken or of citations attracted are intrinsically discrete, and extending our theoretical analysis to deal with ties is potentially of interest.

relations.

The relation R defined on the set $\mathcal{N}$ induces a natural bijective mapping, ρ, of the elements of $\mathcal{N}$ into the set of the first N natural numbers $\rho : \mathcal{N} \longrightarrow \{1, 2, \ldots, N\}$, which satisfies $\rho(x) < \rho(y)$ if and only if xRy. The mapping is the *R-ranking* of the set $\mathcal{N}$, and $\rho(x)$ is simply the rank of x according to the metric induced by R in $\mathcal{N}$: if $\rho(x) < \rho(y)$, then x has a better[6] rank than y. This mapping depends of course on the set $\mathcal{N}$ and on the relation R. As there is no danger of ambiguity, we abuse the notation slightly and leave this dependence implicit. By the same token, we refer to *R-ranking* simply as "ranking" when no ambiguity can arise. As an example, in the set of all England test cricketers, if x is Alec Stewart and y is Nassar Hussain, and the relation R is "has scored more test runs than", then $\rho(x) < \rho(y)$. For another example, in the set of all full professors of organic chemistry in post in an Italian university on 31 December 2010, if the relation R is "has more Google Scholar citations on January 30, 2016", then $\rho(\text{Raffaele Riccio}) < \rho(\text{Marco d'Ischia})$.

We next define a selection $\mathcal{K}$ as a proper and non-empty subset of $\mathcal{N}$.[7] Let $K \in \{1, \ldots, N-1\}$ be the number of elements of $\mathcal{K}$. We define the pair $(\mathcal{N}, \mathcal{K})$ a "selected set". Let $\mathscr{S}$ be the family of selected sets.

We single out the situations where the selection follows exactly the rank-

[6] We use the terminology "best" and "worst" ranked element, rather than highest and lowest, given the potential linguistic ambiguity due to the lowest number being attached to the highest ranked element.

[7] Selecting members of a set is related but different from the administration of a test. In the former, the number of available places is usually fixed, or at least within a given range; whereas the important feature of a test, such as a school leaving exam, or the rejection of potentially faulty items from a production line, is the minimisation of errors, possibly weighted by the relative importance of type I and type II errors. Loosely speaking, one can think of selection according to a metric and testing as inverse operations: in the former, the number of slots is fixed, and the distribution of the metric in the population determines the threshold for selection; in the latter, the pass/fail threshold is fixed, and the distribution determines the number of successful elements.

ing.

Definition 1 *Given a selected set* $(\mathcal{N}, \mathcal{K}) \in \mathscr{S}$*, the selection* $\mathcal{K}$ *is "*PERFECT*" if* $x \in \mathcal{K}$ *and* $y \in \mathcal{N} \setminus \mathcal{K}$ *implies* xRy*. The selection* $\mathcal{K}$ *is "*ANTIPERFECT*" if* $x \in \mathcal{K}$ *and* $y \in \mathcal{N} \setminus \mathcal{K}$ *implies* yRx*.*

In words, the selection $\mathcal{K}$ is PERFECT if no selected element has a rank worse than a non-selected element, and it is ANTIPERFECT if every selected element has a worse rank than every non-selected element.

Let $\mathscr{S}^P \subseteq \mathscr{S}$ be the set of all PERFECT selections, and conversely, let $\mathscr{S}^A \subseteq \mathscr{S}$ be the set of all ANTIPERFECT selections. Note that since $\mathcal{K}$ is a proper non-empty subset of $\mathcal{N}$, $\mathscr{S}^P \cap \mathscr{S}^A = \varnothing$: no selection is simultaneously both PERFECT and ANTIPERFECT.

Next we define the union of the power sets of $\{1, \ldots, N\}$, for $N > 1$, having excluded from each of these power sets the set $\{1, \ldots, N\}$ itself:

$$\mathfrak{S} = \left(\bigcup_{N \in \mathbb{N} \setminus \{1\}} \left(2^{\{1,\ldots,N\}} \setminus \{1, \ldots, N\} \right) \right) \setminus \varnothing.$$

The generic element of the set $\mathfrak{S}$ is $\{i_1, \ldots, i_K\}_N$, the subscript distinguishing identical subsets of the natural numbers when selected from sets with different cardinality. We can now define a function:

$$m : \mathscr{S} \longrightarrow \mathfrak{S},$$

such that for a set $(\mathcal{N}, \mathcal{K}) \in \mathscr{S}$, with N elements, from which K are selected, $m(\mathcal{N}, \mathcal{K})$ is the subset of $\{1, \ldots, N\}$ which are the R-ranks in $\mathcal{N}$ of the elements of $\mathcal{K}$:

$$m : (\mathcal{N}, \mathcal{K}) \longmapsto \left\{ i \in \{1, \ldots, N\} \,|\, \rho^{-1}(i) \in \mathcal{K} \right\}.$$

Given a selected set $(\mathcal{N}, \mathcal{K})$, we may represent its image under m as follows:

$$(\mathcal{N}, \mathcal{K}) \longrightarrow 011000010001000. \tag{1}$$

Elements are ranked from best to the left, to worst to the right, and a "1" in the j-th position indicates that the j-th element is selected. Thus, in (1), the second, the third, the eighth and the twelfth R-ranked elements are selected. Note that this representation distinguishes the selection of the same ranked elements from sets of different cardinality.

We want to compare selections, that is order the set $\mathscr{S}$, in the following sense: consider two ranked sets $\mathcal{N}_A$ with N_A elements ranked by R_A, and $\mathcal{N}_B$ with N_B elements ranked by R_B, and selections $\mathcal{K}_A$ and $\mathcal{K}_B$ from $\mathcal{N}_A$ and $\mathcal{N}_B$, respectively. We want to answer the question: is the selection $\mathcal{K}_A$ from $\mathcal{N}_A$ closer to or farther from the ranking of set $\mathcal{N}_A$ induced by R_A than the selection $\mathcal{K}_B$ from $\mathcal{N}_B$ is to the ranking of set $\mathcal{N}_B$ induced by R_B? In terms of the examples given above, we want to know whether, say, a journalist's choice of the "all time England cricket test team" is more based on the players' record than an Italian chemistry academy's choice of the members of its scientific committee is based on the academics' citation count.

To formalise this question we define a binary relation $\mathfrak{M} \subseteq \mathfrak{S} \times \mathfrak{S}$, which we interpret as "*rankiness*": rankiness is the property of being close to the ranking of the set.

Definition 2 *Given two selected sets* $(\mathcal{N}_A, \mathcal{K}_A), (\mathcal{N}_B, \mathcal{K}_B) \in \mathscr{S}$ *with relations* R_A *and* R_B*,* $(\mathcal{N}_A, \mathcal{K}_A)$ *is at least as rank-based as* $(\mathcal{N}_B, \mathcal{K}_B)$ *if and only if* $\big(m(\mathcal{N}_A, \mathcal{K}_A), m(\mathcal{N}_B, \mathcal{K}_B)\big) \in \mathfrak{M}$*.*

We are thus defining equivalence classes in $\mathscr{S}$: two selected sets $(\mathcal{N}_A, \mathcal{K}_A)$, $(\mathcal{N}_B, \mathcal{K}_B) \in \mathscr{S}$ are in the same equivalence class if and only if $m(\mathcal{N}_A, \mathcal{K}_A) =$

$m(\mathcal{N}_B, \mathcal{K}_B)$, that is if the sets $\mathcal{N}_A$ and $\mathcal{N}_B$ have the same number of elements, and the ranks of the selected elements are the same in the selected sets $(\mathcal{N}_A, \mathcal{K}_A)$ and $(\mathcal{N}_B, \mathcal{K}_B)$. Thus Definition 2, accordingly, defines rankiness as a relation on the classes of equivalence in $\mathscr{S}$.

We follow the standard convention used to describe preferences, and write $(\mathcal{N}_A, \mathcal{K}_A) \succsim_{\mathfrak{M}} (\mathcal{N}_B, \mathcal{K}_B)$ when $(\mathcal{N}_A, \mathcal{K}_A)$ is at least as rank-based as $(\mathcal{N}_B, \mathcal{K}_B)$. "Strict rankiness", $\succ_{\mathfrak{M}}$, and "equal rankiness", $\sim_{\mathfrak{M}}$, are naturally defined: $(\mathcal{N}_A, \mathcal{K}_A)$ is strictly more rank-based than $(\mathcal{N}_B, \mathcal{K}_B)$ if $(\mathcal{N}_A, \mathcal{K}_A) \succsim_{\mathfrak{M}} (\mathcal{N}_B, \mathcal{K}_B)$ and not $(\mathcal{N}_B, \mathcal{K}_B) \succsim_{\mathfrak{M}} (\mathcal{N}_A, \mathcal{K}_A)$. And $(\mathcal{N}_A, \mathcal{K}_A)$ and $(\mathcal{N}_B, \mathcal{K}_B)$ are equally rank-based if $(\mathcal{N}_A, \mathcal{K}_A) \succsim_{\mathfrak{M}} (\mathcal{N}_B, \mathcal{K}_B)$ and $(\mathcal{N}_B, \mathcal{K}_B) \succsim_{\mathfrak{M}} (\mathcal{N}_A, \mathcal{K}_A)$.

We require the rankiness relation $\mathfrak{M}$ to be reflexive, so that all selected sets with the same image are equally rank-based, complete and transitive. The assumption of completeness is a strong one, though it is necessary to ensure that rankiness can have operational value in practice, as it ensures that there are no selections which are "not comparable". Relatively simple examples show that it is in principle arbitrary to construct such a complete ordering. Consider two selected sets $(\mathcal{N}_A, \mathcal{K}_A), (\mathcal{N}_B, \mathcal{K}_B) \in \mathscr{S}$, and let them be represented graphically as:

$$(\mathcal{N}_A, \mathcal{K}_A) \longrightarrow 0100111001000001000000010001000000,$$
$$(\mathcal{N}_B, \mathcal{K}_B) \longrightarrow 001001010110000000000.$$

In $(\mathcal{N}_A, \mathcal{K}_A)$, there are some selected elements among the best ranked, but there are also some below the median. In the second, from a smaller set, selected elements are all above the median, but many are close to it: different observers might well have different views as to which of the two above selections is more rank-based. As we show in this paper, this arbitrariness is fully resolved if one accepts the simple axioms we propose.

Even with completeness, without any further restrictions, the rankiness relation can still be vacuous: for example, the relation $\mathfrak{M} = \mathfrak{S} \times \mathfrak{S}$, where all selected sets are equally rank-based is transitive, reflexive and complete. In the rest of the paper, therefore, we impose some further requirements. As in BBP, these axioms impose natural requirements of the relation between R, the relation among elements of the sets $\mathcal{N}$, and $\mathfrak{M}$, the relation between the images in $\mathfrak{S}$ of the sets $\mathcal{N}$. A natural requirement imposed by BBP (p 904) is the "extension rule": given $x, y \in \mathcal{N}$, then $m(\mathcal{N}, \{x\}) \succsim_{\mathfrak{M}} m(\mathcal{N}, \{y\})$ if and only if xRy. Note that we omit the subscript N, writing, for example $\{x\}$ for $\{x\}_N$, when no confusion can possibly arise. We strengthen the extension rule, requiring the rankiness comparison to be strict, and applying it to any set, not just singletons.

Axiom 1 (Swap-Dominance) *For all* $(\mathcal{N}, \mathcal{K}) \in \mathscr{S}$ *and* $x \in \mathcal{N} \backslash \mathcal{K}$ *and* $y \in \mathcal{K}$, $(\mathcal{N}, \mathcal{K} \cup \{x\} \setminus \{y\}) \succ_{\mathfrak{M}} (\mathcal{N}, \mathcal{K})$ *if and only if* xRy.

In words, Axiom 1 requires that the swap between an element in the selection and an element not in the selection makes the selected set strictly more (less) rank-based if the rank of the newly selected element is better (worse) than the rank of the removed element.[8] Note that Axiom 1 is incompatible with the independence axiom (BBP p 905), which in our framework, would require that given selections $(\mathcal{N}, \mathcal{K}_{\mathcal{A}}), (\mathcal{N}, \mathcal{K}_{\mathcal{B}}) \in \mathscr{S}$ and $x \in \mathcal{N} \setminus (\mathcal{K}_{\mathcal{A}} \cup \mathcal{K}_{\mathcal{B}})$, $(\mathcal{N}, \mathcal{K}_{\mathcal{A}}) \succ_{\mathfrak{M}} (\mathcal{N}, \mathcal{K}_{\mathcal{B}})$ implies $(\mathcal{N}, \mathcal{K}_{\mathcal{A}} \cup \{x\}) \succsim_{\mathfrak{M}} (\mathcal{N}, \mathcal{K}_{\mathcal{B}} \cup \{x\})$. As they note, this axiom rules out "certain types of complementarities" (p 906), and runs therefore contrary to the motivation of the paper, which views selections

[8]One could make an analogy with the Dalton-Pigou principle (Dalton 1920, p 351); a transfer of a resource (being selected in our case, or income in Dalton's) from a worse ranked/richer to a better ranked/poorer element/person, so long as that transfer does not reverse the ranking of the two, will result in greater rankiness/equity.

in their entirety. In fact, BBP show that independence is a strong requirement in this context, as it prevents comparison between selections, except in very special cases (BBP, pp 910-922).

The next two Axioms we impose are the natural extension of the idea of dominance, which "requires that adding an element which is better (worse) than all elements in a given set A according to R leads to a set that is better (worse) than the original set" (BBP, p 905). In our more complex set-up, we want to compare subsets selected from different sets, and therefore we state the axioms as binary comparisons between sets with different number of elements.

Axiom 2 (Better-Dominance) *For all* $(\mathcal{N}, \mathcal{K}) \in \mathscr{S}$ *and* $x \notin \mathcal{N}$ *such that* xRy *for all* $y \in \mathcal{K}$*:*

i. $(\mathcal{N} \cup \{x\}, \mathcal{K} \cup \{x\}) \succsim_{\mathfrak{M}} (\mathcal{N}, \mathcal{K})$*; strictly unless* $(\mathcal{N}, \mathcal{K}) \in \mathscr{S}^P$.

ii. $(\mathcal{N}, \mathcal{K}) \succsim_{\mathfrak{M}} (\mathcal{N} \cup \{x\}, \mathcal{K})$*; strictly unless* $(\mathcal{N}, \mathcal{K}) \in \mathscr{S}^A$.

Suppose a new element is added[9] to the set $\mathcal{N}$, and this new element has better rank than every selected element. Then Axiom 2 requires that, if this new element is selected, the selection becomes more rank-based (Axiom 2.i); if it is not selected, the selection becomes less rank-based (Axiom 2.ii).

Axiom 3 (Worse-Dominance) *For all* $(\mathcal{N}, \mathcal{K}) \in \mathscr{S}$ *and* $x \notin \mathcal{N}$ *such that* yRx *for all* $y \in \mathcal{K}$*:*

i. $(\mathcal{N} \cup \{x\}, \mathcal{K}) \succsim_{\mathfrak{M}} (\mathcal{N}, \mathcal{K})$*; strictly unless* $(\mathcal{N}, \mathcal{K}) \in \mathscr{S}^P$.

ii. $(\mathcal{N}, \mathcal{K}) \succsim_{\mathfrak{M}} (\mathcal{N} \cup \{x\}, \mathcal{K} \cup \{x\})$*; strictly unless* $(\mathcal{N}, \mathcal{K}) \in \mathscr{S}^A$.

[9]Note that the statement of Axioms 2 and 3 requires that x can be put in the relation R with the existing elements of $\mathcal{N}$. Thus if $\mathcal{N}$ is the set of English test cricketers, x is a newly eligible player; if $\mathcal{N}$ is the set of Italian chemistry professors, x is a newly appointed one.

Axiom 3 is the converse of Axiom 2 at the other end of the ranking of the set $\mathcal{N}$: if a new element is added to the set $\mathcal{N}$ which is worse ranked than every selected element, then the selection becomes more (respectively, less) rank-based if this new element is not selected, Axiom 3.i (respectively, is selected, Axiom 3.ii).

To see these axioms "in action", consider the selected set represented in (1). Axioms 2 and 3 are illustrated by the following changes in the selected set, where the new element is boxed.

Axiom	new element in set (1)	selected Y/N?	New selected set	more/less rank-based?
2.i	better than every selected	yes	$\boxed{1}011000010001000$	more
2.ii	better than every selected	no	$0\boxed{0}11000010001000$	less
3.ii	worse than every selected	yes	$01100001000100\boxed{1}0$	less
3.i	worse than every selected	no	$011000010001\boxed{0}000$	more

In the first and fourth rows the selection becomes more rank-based, and in the second and third less so. In the first two rows the new element can indifferently be in the first or second position of the ranking; and in the third and fourth row, in any of the bottom four positions.

The selection (1) is neither PERFECT nor ANTIPERFECT, and so all the comparisons between it and the selected sets in the above table are strict.

Axioms 2 and 3 dictate the relative rankiness of two selected sets where the ranked sets, $\mathcal{N}$ and $\mathcal{N} \cup \{x\}$, differ in size by 1. The dominance axioms, labelled by BBP (p 905) the Gärdenfors principle (Gärdenfors 1976), impose an ordering on selections from a given ranked set (hence in our framework consider sets of the same size). We derive them in our framework as immediate consequences of Axioms 2 and 3 in the following corolllary.

Corollary 1 *i. For all* $(\mathcal{N}, \mathcal{K}) \in \mathscr{S}$ *and* $x \in \mathcal{N} \backslash \mathcal{K}$ *such that* xRy *for all*

$y \in \mathcal{K}$, $(\mathcal{N}, \mathcal{K} \cup \{x\}) \succsim_{\mathfrak{M}} (\mathcal{N}, \mathcal{K})$; strictly unless $(\mathcal{N}, \mathcal{K}) \in \mathscr{S}^A$.

ii. for all $(\mathcal{N}, \mathcal{K}) \in \mathscr{S}$ and $x \in \mathcal{K}$ such that xRy for all $y \in \mathcal{K} \setminus \{x\}$, $(\mathcal{N}, \mathcal{K}) \succsim_{\mathfrak{M}} (\mathcal{N}, \mathcal{K} \setminus \{x\})$; strictly unless $(\mathcal{N}, \mathcal{K}) \in \mathscr{S}^P$.

iii. For all $(\mathcal{N}, \mathcal{K}) \in \mathscr{S}$ and $x \in \mathcal{K}$ such that yRx for all $y \in \mathcal{K} \setminus \{x\}$, $(\mathcal{N}, \mathcal{K} \setminus \{x\}) \succsim_{\mathfrak{M}} (\mathcal{N}, \mathcal{K})$; strictly unless $(\mathcal{N}, \mathcal{K}) \in \mathscr{S}^P$.

iv. for all $(\mathcal{N}, \mathcal{K}) \in \mathscr{S}$ and $x \in \mathcal{N} \setminus \mathcal{K}$ such that yRx for all $y \in \mathcal{K}$, $(\mathcal{N}, \mathcal{K}) \succsim_{\mathfrak{M}} (\mathcal{N}, \mathcal{K} \cup \{x\})$; strictly unless $(\mathcal{N}, \mathcal{K}) \in \mathscr{S}^A$.

Proof. We only establish the first claim, as the proof of the remainder is essentially identical. Consider a selected set $(\mathcal{N}, \mathcal{K})$, and an $x \in \mathcal{N} \setminus \mathcal{K}$ such that xRy for all $y \in \mathcal{K}$. Note that this implies that $(\mathcal{N}, \mathcal{K}) \notin \mathscr{S}^P$. Take $z \notin \mathcal{N}$, such that zRy for all $y \in \mathcal{K}$, and xRz (that is z is better than every selected element, but worse than x). By Axiom 2.i, $(\mathcal{N}, \mathcal{K}) \succ_{\mathfrak{M}} (\mathcal{N} \cup \{z\}, \mathcal{K})$. Next note that, by Axiom 3.ii, $(\mathcal{N} \cup \{z\}, \mathcal{K}) \succ_{\mathfrak{M}} (\mathcal{N} \cup \{z\} \setminus \{x\}, \mathcal{K})$. Finally, notice that $m(\mathcal{N} \cup \{z\} \setminus \{x\}, \mathcal{K}) = m(\mathcal{N} \cup \{x\}, \mathcal{K})$, since an excluded element better than all selected elements is replaced by another, and so $(\mathcal{N}, \mathcal{K}) \succ_{\mathfrak{M}} (\mathcal{N} \cup \{x\}, \mathcal{K})$. ■

Intuitively, if a non-selected element that is better ranked than every selected element were instead selected, the selection would become more rank-based (Corollary 1.i), and if the best element in the selection were removed, the selection would become less rank-based (Corollary 1.ii). Conversely, consider an element in $\mathcal{N}$ which is worse ranked than every selected element: if it is removed from the selection, then the selection becomes more rank-based (Corollary 1.iii); if it is added to the selection, then the selection becomes less rank-based (Corollary 1.iv).

Taken together, the three axioms deal with different segments of the rankings: Axioms 2 and 3 add a new element, at the beginning and the end of the ranking, and Axiom 1 swaps two elements, which can therefore be anywhere. Thus they are independent: it is possible to define different relations $\mathfrak{M}_1$, $\mathfrak{M}_2$, and $\mathfrak{M}_3$ on $\mathfrak{S} \times \mathfrak{S}$ such that each is violated when the other two are respected.

While Corollary 1 is an obvious consequence of Axioms 2 and 3, the next result is less immediate, and shows that, although Axioms 2-1 may seem innocuous, they do have some bite, as can be inferred from one of their implications on the comparison of "extreme" selections from different sized sets.

Proposition 1 *i. Let $(\mathcal{N}_{p_1}, \mathcal{K}_{p_1}), (\mathcal{N}_{p_2}, \mathcal{K}_{p_2}) \in \mathscr{S}^P$. Then $(\mathcal{N}_{p_1}, \mathcal{K}_{p_1}) \sim_{\mathfrak{M}} (\mathcal{N}_{p_2}, \mathcal{K}_{p_2})$.*

ii. Similarly, let $(\mathcal{N}_{a_1}, \mathcal{K}_{a_1}), (\mathcal{N}_{a_2}, \mathcal{K}_{a_2}) \in \mathscr{S}^A$. Then $(\mathcal{N}_{a_1}, \mathcal{K}_{a_1}) \sim_{\mathfrak{M}} (\mathcal{N}_{a_2}, \mathcal{K}_{a_2})$.

iii. Let $(\mathcal{N}, \mathcal{K}) \in \mathscr{S} \backslash \mathscr{S}^P \backslash \mathscr{S}^A$; let $(\mathcal{N}_p, \mathcal{K}_p) \in \mathscr{S}^P$; let $(\mathcal{N}_a, \mathcal{K}_a) \in \mathscr{S}^A$. Then $(\mathcal{N}_p, \mathcal{K}_p) \succ_{\mathfrak{M}} (\mathcal{N}, \mathcal{K})$ and $(\mathcal{N}, \mathcal{K}) \succ_{\mathfrak{M}} (\mathcal{N}_a, \mathcal{K}_a)$.

Proof. Let us begin with (i). Consider a PERFECT selection of K_p elements from a set $(\mathcal{N}_p, \mathcal{K}_p)$, with $(\mathcal{N}_p, \mathcal{K}_p) \in \mathscr{S}^P$. Clearly the best ranked K_p elements are in $\mathcal{K}_p$, the rest in $\mathcal{N}_p \backslash \mathcal{K}_p$. Now add to the set and to the selection an element z_1 such that $z_1 R x$ for all $x \in \mathcal{K}_p$. By Axiom 2.i, $(\mathcal{N}_p, \cup \{z_1\}, \mathcal{K}_p \cup \{z_1\}) \succsim_{\mathfrak{M}} (\mathcal{N}_p, \mathcal{K}_p)$. Next return to the selection $(\mathcal{N}_p, \mathcal{K}_p)$, and add, again to the set and to the selection, an element z_2 such that $x R z_2$ for all $x \in \mathcal{K}_p$, and $z_2 R y$ for all $y \in \mathcal{N}_p \backslash \mathcal{K}_p$ (that is z_2 is worse ranked than every selected element, but better ranked than every non-selected element). By Axiom 3.ii, $(\mathcal{N}_p, \mathcal{K}_p) \succsim_{\mathfrak{M}} (\mathcal{N}_p, \cup \{z_2\}, \mathcal{K}_p \cup \{z_2\})$. But now notice that both new selections select only the best ranked $K_p + 1$ elements: hence, $m(\mathcal{N}_p, \cup \{z_1\}, \mathcal{K}_p \cup \{z_1\}) = m(\mathcal{N}_p, \cup \{z_2\}, \mathcal{K}_p \cup \{z_2\})$. By transitivity, these both equal $m(\mathcal{N}_p, \mathcal{K}_p)$. The process can be repeated to show that all PERFECT selections are equally rank-based. The proof of (ii) is identical. Consider (iii) next.

Take a selection $(\mathcal{N}, \mathcal{K})$ with $(\mathcal{N}, \mathcal{K}) \in \mathscr{S} \backslash \mathscr{S}^P \backslash \mathscr{S}^A$. Let N and K the number of elements in $\mathcal{N}$ and $\mathcal{K}$. Now let $x_1 \in \mathcal{K}$ be the best ranked selected element such that there is a non-selected element $y \in \mathcal{N} \backslash \mathcal{K}$ such that yRx_1. Because the selection is not PERFECT, it is possible to find such a x_1. Next let $y_1 \in \mathcal{N} \backslash \mathcal{K}$ be the best ranked non-selected element such that there a selected element $x \in \mathcal{K}$ such that $y_1 Rx$. Again, because the selection is not ANTIPERFECT, it is possible to find such a y_1. Now by Axiom 1, $(\mathcal{N}, \mathcal{K} \cup \{x_1\} \setminus \{y_1\}) \succ_{\mathfrak{M}} (\mathcal{N}, \mathcal{K})$. If the selection $(\mathcal{N}, \mathcal{K} \cup \{x_1\} \setminus \{y_1\})$ is PERFECT, then we are done. If not, we can repeat, until a perfect selection in reached. This happens in at most $\min \{K, N - K\}$ steps, and establishes that $(\mathcal{N}_p, \mathcal{K}_p) \succ_{\mathfrak{M}} (\mathcal{N}, \mathcal{K})$. The demonstration that $(\mathcal{N}, \mathcal{K}) \succ_{\mathfrak{M}} (\mathcal{N}_a, \mathcal{K}_a)$ is identical. ■

In words, Proposition 1 says that all PERFECT selections are equally rank-based, and similarly, that all ANTIPERFECT selections are equally rank-based. The third statement asserts that every PERFECT selection is strictly more rank-based than every non-PERFECT selection, and every ANTIPERFECT selection is strictly less rank-based than every non-ANTIPERFECT selection. This is entirely reasonable when the selection is from the same set. Thus, selecting the best ten from a set of one hundred elements is clearly more rank-based than selecting the best nine and the eleventh. However, as we want to extend the range of selected sets to be compared, it is possible to think of situations when the size is of the sets and of the selections is different, and when the argument that a PERFECT selection is more rank-based is less clear cut. Consider, for example, the following extreme case: is selecting the better of the two elements of a set unquestionably more rank-based than selecting the best 25 and the 27-th ranked out of a set with 10,000 elements? Proposition 1 answers unambiguously yes, but someone might argue in favour of the opposite, on the grounds that the former is more likely than the latter to be determined by other criteria, which by chance happen to coincide with the metric, whereas the selection of the first 25 and the 27-th elements out of a very large set would almost surely be seen as the consequence of a determinate intention to use the metric as the criterion for selection, with the 26-th element being excluded due to some other criterion.

3 An index of "rankiness"

Definition 3 *An* index of rankiness *is a function* $M : \mathfrak{S} \longrightarrow [-1, 1]$ *such that, given any two selected sets* $(\mathcal{N}_A, \mathcal{K}_A), (\mathcal{N}_B, \mathcal{K}_B) \in \mathscr{S}$, $(\mathcal{N}_A, \mathcal{K}_A) \succsim_{\mathfrak{M}} (\mathcal{N}_B, \mathcal{K}_B)$ *if and only if* $M(m(\mathcal{N}_A, \mathcal{K}_A)) \geqslant M(m(\mathcal{N}_B, \mathcal{K}_B))$.

In analogy with the theory of consumers' preferences, existence of the index of rankiness is ensured by the additional requirement that the rankiness relation is continuous. The formal proof is identical with the proof of the existence of a utility function; as with consumer theory, a lexicographic relation illustrates well the need of continuity to represent rankiness through an index. Consider the following way of comparing the rankiness of any two selected sets: take first the best ranked element in each: if only one selection includes it, then that selection is strictly more rank-based than the other. Otherwise, look next at the second best ranked element, and again, if it is included only in one selection, this is the more rank-based one. Again, if they are both or neither selected, go to the next ranked element and so on. If one selection "runs out of elements" before the other, then it is less rank-based. In analogy with consumer preferences, this relation is reflexive, complete and transitive, but it is not continuous, and so it cannot be represented by an index of rankiness.[10]

[10] The relation given in the text can be formalised as follows.

To ensure existence of a rankiness index representing a relation $\mathfrak{M}$, therefore we require that the rankiness relation be continuous.[11]

Uniqueness, on the other hand, is not ensured by the three axioms proposed: it is possible to find pairs of relations both satisfying the three axioms, which rank differently the rankiness of given selected sets, and therefore are represented by indices of rankiness which attach differently ordered values to given selected sets. To see this, consider the following two indices.

$$M\left(m\left(\mathcal{N},\mathcal{K}\right)\right)=\frac{N+1-\frac{2}{K}\sum_{x\in\mathcal{K}}\rho\left(x\right)}{N-K},\tag{2}$$

and

$$M_1(m\left(\mathcal{N},\mathcal{K}\right))=\frac{\frac{2K^2+1}{3}+N\left(N-K+1\right)-\frac{2}{K}\sum_{x\in\mathcal{K}}\rho\left(x\right)^2}{\left(N+1\right)\left(N-K\right)}.\tag{3}$$

For future reference, it is convenient to denote the sum of the ranks of the selected elements as:

$$r=\sum_{x\in\mathcal{K}}\rho\left(x\right).\tag{4}$$

Given any two selected sets $(\mathcal{N}_A,\mathcal{K}_A),(\mathcal{N}_B,\mathcal{K}_B)\in\mathscr{S}$, take $x_1\in\mathcal{N}_A$ with $\rho(x_1)=1$ and $y_1\in\mathcal{N}_B$ with $\rho(y_1)=1$. If $x_1\in\mathcal{K}_A$ and $y_1\notin\mathcal{K}_B$ then $(\mathcal{N}_A,\mathcal{K}_A)\succ_{\mathfrak{M}}(\mathcal{N}_B,\mathcal{K}_B)$, and *vice versa* if $x_1\notin\mathcal{K}_A$ and $y_1\in\mathcal{K}_B$ then $(\mathcal{N}_B,\mathcal{K}_B)\succ_{\mathfrak{M}}(\mathcal{N}_A,\mathcal{K}_A)$. Otherwise, that is if either ($x_1\in\mathcal{K}_A$ and $y_1\in\mathcal{K}_B$) or ($x_1\notin\mathcal{K}_A$ and $y_1\notin\mathcal{K}_B$), then take $x_2\in\mathcal{N}_A$ with $\rho(x_2)=2$ and $y_2\in\mathcal{N}_B$ with $\rho(y_2)=2$. If $x_2\in\mathcal{K}_A$ and $y_2\notin\mathcal{K}_B$ then $(\mathcal{N}_A,\mathcal{K}_A)\succ_{\mathfrak{M}}(\mathcal{N}_B,\mathcal{K}_B)$, and *vice versa* if $x_2\notin\mathcal{K}_A$ and $y_2\in\mathcal{K}_B$ then $(\mathcal{N}_B,\mathcal{K}_B)\succ_{\mathfrak{M}}(\mathcal{N}_A,\mathcal{K}_A)$.
For $i\geqslant 2$: if ($x_i\in\mathcal{K}_A$ and $y_i\in\mathcal{K}_B$) or if ($x_i\notin\mathcal{K}_A$ and $y_i\notin\mathcal{K}_B$), then: if $K_A=i$ and $K_B=i$, then $(\mathcal{N}_B,\mathcal{K}_B)\sim_{\mathfrak{M}}(\mathcal{N}_A,\mathcal{K}_A)$; if $K_A>i$ and $K_B=i$, then $(\mathcal{N}_A,\mathcal{K}_A)\succ_{\mathfrak{M}}(\mathcal{N}_B,\mathcal{K}_B)$, and *vice versa* if $K_A=i$ and $K_B>i$, then $(\mathcal{N}_B,\mathcal{K}_B)\succ_{\mathfrak{M}}(\mathcal{N}_A,\mathcal{K}_A)$; if $K_A>i$ and $K_B>i$, then take $x_{i+1}\in\mathcal{N}_A$ with $\rho(x_{i+1})=i+1$ and $y_{i+1}\in\mathcal{N}_B$ with $\rho(y_{i+1})=i+1$. If $x_{i+1}\in\mathcal{K}_A$ and $y_{i+1}\notin\mathcal{K}_B$ then $(\mathcal{N}_A,\mathcal{K}_A)\succ_{\mathfrak{M}}(\mathcal{N}_B,\mathcal{K}_B)$, and *vice versa* if $x_{i+1}\notin\mathcal{K}_A$ and $y_{i+1}\in\mathcal{K}_B$ then $(\mathcal{N}_B,\mathcal{K}_B)\succ_{\mathfrak{M}}(\mathcal{N}_A,\mathcal{K}_A)$.

[11] That is, the inverse image $\mathfrak{M}$ of any open subset of $\mathfrak{S}$ is itself open. Given a set $\mathfrak{s}\subseteq\mathfrak{S}$, its inverse image is the set $\mathfrak{M}^{-1}(\mathfrak{s})=\{s\in\mathfrak{S}|\exists s_1\in\mathfrak{s}:s\mathfrak{M}s_1\}$. Because the cardinality of the set $\mathfrak{S}$ is the same as that of the set of real numbers (Lucas 1990, pp 134–135), open sets in $\mathfrak{S}$ are those that are put in correspondence with open intervals in $\mathbb{R}$ by a surjection.

Using (4), the expression in (2) can then be written as

$$M(m(\mathcal{N},\mathcal{K})) = \frac{r^{N,K} + r_K - 2r}{r^{N,K} - r_K},$$

where r_K and $r^{N,K}$ are the sum of the ranks when the best and the worst K elements are selected, respectively, and therefore they are given by: $r_K = \sum_{j=1}^{K} j$, and $r^{N,K} = \sum_{j=N-K+1}^{N} j$. Hence (2) takes value 1 for a PERFECT selection, and value -1 for an ANTIPERFECT one. Similarly for $M_1(m(\mathcal{N},\mathcal{K}))$: it can be written as

$$M_1(m(\mathcal{N},\mathcal{K})) = \frac{r_1^{N,K} + r_{1,K} - 2\sum_{x\in\mathcal{K}} \rho(x)^2}{r_1^{N,K} - r_{1,K}},$$

where $r_{1,K}$ and $r_1^{N,K}$ are the sum of the *squares* of the ranks when the best and the worst K elements are selected, $r_{1,K} = \sum_{j=1}^{K} j^2$, and $r_1^{N,K} = \sum_{j=N-K+1}^{N} j^2$.

Proposition 2 *A relation represented by the index $M(m(\mathcal{N},\mathcal{K}))$ satisfies Axioms 2-1.*

Proof. Suppose a relation $\mathfrak{M}$ on $\mathfrak{S}\times\mathfrak{S}$ is given, which can be represented by the index (2), $M(m(\mathcal{N},\mathcal{K}))$. We begin by showing that $\mathfrak{M}$ satisfies Axioms 2 and 3. In each of these axioms, a new element z is added to the set $\mathcal{N}$. Therefore N increases by 1. What happens to K and to the total rank r depends on which part of which Axiom is considered. In Axiom 2.i the new element z is in the selection, and so K also increases by 1, and as z is better than every selected element, the total rank increases by $\rho(z)+K$: each of the K previously selected elements increases by 1, and the new element's rank $\rho(z)$ is added to the total. The value of M therefore changes from

$$\frac{N+1-\frac{2}{K}r}{N-K} \tag{5}$$

to

$$\frac{N+2-\frac{2}{K+1}\left(r+\rho\left(z\right)+K\right)}{N-K}.$$

The difference is

$$\frac{2\frac{r}{K}-2\rho\left(z\right)-K+1}{\left(N-K\right)\left(K+1\right)}, \tag{6}$$

which is increasing in r, and equals $\frac{2}{(N-K)(K+1)}>0$ when r takes its lowest possible value given $\rho\left(z\right)$, $\sum_{j=\rho(z)+1}^{\rho(z)+K} j$. Thus it is positive for every feasible value of r.

If the new element is not selected, Axiom 2.ii, then K does not change, but the total rank of the selected elements increases by K, since the rank of every selected element increases by 1, and so the value of the index changes from (5) to

$$\frac{N-\frac{2}{K}r}{N+1-K}.$$

The difference with (5) is $-\frac{2N-K+1-\frac{2}{K}r}{(N+1-K)(N-K)}$ which is increasing in r, and since it is 0 at the maximum value of r, $r^{N,K}=\frac{K(2N-K+1)}{2}$, it is strictly negative for any other value of r, establishing the result.

Next consider Axiom 3. There is a new element in $\mathcal{N}$ which is worse than all the elements selected. In Axiom 3.i, the new element is not selected, and so neither K nor r change. In this case, the index (2) becomes

$$\frac{N+2-\frac{2}{K}r}{N+1-K}.$$

The difference with its previous value, (5), is $-\frac{K+1-\frac{2}{K}r}{(N+1-K)(N-K)}$, which is increasing in r. Since it is 0 at the minimum value of r, which is $r_K=\frac{K(K+1)}{2}$, it is strictly positive for any other value of r, and so the index increases in this case. Finally Axiom 3.ii: the new element, worse than all those already selected, is itself selected. Thus K increases by 1 and r by $\rho\left(z\right)$, and so index (2) becomes

$$\frac{N+2-\frac{2}{K+1}\left(r+\rho\left(z\right)\right)}{N-K},$$

Table 1: Summary of the proof of Proposition 2.

$\forall y \in \mathcal{K}, z \notin \mathcal{N}, zRy$	$\forall y \in \mathcal{K}, z \notin \mathcal{N}, yRz$
Axiom 2.i: $z \in \mathcal{K}$. r increases by $\rho(z) + K$, K increases by 1. The new value of M is higher: $\frac{N+2-\frac{2}{K+1}(r+\rho(z)+K)}{N-K} > M$	Axiom 3.ii: $z \in \mathcal{K}$. r increases by $\rho(z)$, K increases by 1. The new value of M is lower: $\frac{N+2-\frac{2}{K+1}(r+\rho(z))}{N-K} < M$
Axiom 2.ii: $z \notin \mathcal{K}$. r increases by K, K does not change. The new value of M is lower: $\frac{N-\frac{2}{K}r}{N+1-K} < M$	Axiom 3.i: $z \notin \mathcal{K}$. r does not change, K does not change. The new value of M is higher: $\frac{N+2-\frac{2}{K}r}{N+1-K} > M$

and the difference with (5) is $\frac{K+1-\rho(z)+\frac{2}{K}r}{(N-K)(K+1)}$, increasing in r and taking, for given $\rho(z)$, its maximum value, 0, at $r = \sum_{j=\rho(z)-K}^{\rho(z)-1} j = K\left(\rho(z) - \frac{1+K}{2}\right)$.

Consider finally Axiom 1, which is straightforward: the swap between an element in the selection and an element not in the selection changes neither K nor N. It only changes r, and so clearly the index M increases if r decreases, that is if a better ranked element takes the place of a worse ranked one in the selection. This establishes that the index (2) satisfies all the Axioms and completes the proof. ∎

Table 1 summarises the proof of Lemma 2, by presenting a schematic description of the effects of adding a new element to a selected set, and of the effects which Axioms 2 and 3 require on the value of the index: in all cases, N increases by 1, and K, r, and M are the original values of the size of the selection, of the sum of the ranks, and of the index of rankiness.

The analogous of Proposition 2 for the index (3) holds, but its proof is essentially identical to the above proof and is omitted.

Proposition 3 *A relation represented by the index* $M_1(m(\mathcal{N}, \mathcal{K}))$ *satisfies*

Axioms 1-3.

Now consider the following two selected sets, which differ only in the selection of the elements in the boxes,

$$\boxed{0}110\boxed{1}001\boxed{1}0\boxed{0},$$
$$\boxed{1}110\boxed{0}001\boxed{0}0\boxed{1}.$$

If the relation of rankiness is represented by index (2), then the second is more rank-based, and vice versa, if rankiness is described by a relation represented by index (3) then the first is more rank-based. In other words, both indices M and M_1 satisfy Axioms 1-3, and yet they give a different answer to the question of the relative rankiness of the two sets: Axioms 1-3 are not characterising.

The restriction required to ensure characterisation is remarkably simple, as the rest of the paper shows. We replace Axiom 1 with the following.

Axiom 4 (Mirror invariance) *For all* $(\mathcal{N},\mathcal{K}) \in \mathscr{S}$ *and* $x \notin \mathcal{N}$, $\left(\mathcal{N} \cup \{x\}, \mathcal{K} \cup \{x\}\right) \succsim_{\mathfrak{M}} (\mathcal{N},\mathcal{K})$ *if and only if* $(\mathcal{N},\mathcal{N}\backslash\mathcal{K}) \succsim_{\mathfrak{M}} (\mathcal{N} \cup \{x\}, \mathcal{N}\backslash\mathcal{K})$.

In words, suppose that a new element is added to the set $\mathcal{N}$, and that this makes the new selected set more rank-based. Then it must be the case that the "mirror image" of the new selected set, that is selection from the same set which includes all the elements which are not selected in the original selection, and excludes all those which were included, is less rank-based than the mirror image of the original selected set. Axiom 4 is stated for the case when the new element is selected, but it of course implies the opposite case: $\left(\mathcal{N} \cup \{x\}, \mathcal{K}\right) \succsim_{\mathfrak{M}} (\mathcal{N},\mathcal{K})$ if and only if $(\mathcal{N},\mathcal{N}\backslash\mathcal{K}) \succsim_{\mathfrak{M}} (\mathcal{N} \cup \{x\}, \mathcal{N}\backslash\mathcal{K} \cup \{x\})$. Axiom

4 is illustrated below, using the representation of a selected set given above:

$$\begin{aligned}
(\mathcal{N},\mathcal{K}) &\longrightarrow 00100101011000000000,\\
(\mathcal{N},\mathcal{N}\backslash\mathcal{K}) &\longrightarrow 11011010100111111111,\\
(\mathcal{N}\cup\{x\},\mathcal{K}\cup\{x\}) &\longrightarrow 0010\boxed{1}0101011000000000,\\
(\mathcal{N}\cup\{x\},\mathcal{N}\backslash\mathcal{K}) &\longrightarrow 1101\boxed{0}1010100111111111.
\end{aligned}$$

If the first selected set is more rank-based than the third, then it must be the case that the second is less rank-based than the fourth, and *vice versa*.

We can now establish the main result of the paper.

Theorem 1 *A reflexive transitive, complete and continuous rankiness relation $\mathfrak{M}$ satisfies Axioms 2, 3 and 4 if and only if it can be represented by a monotonic transformation of the index of rankiness $M(m(\mathcal{N},\mathcal{K}))$, given in (2).*

The "if" part of this result follows immediately from Proposition 2: to see this simply note that Axiom 4 implies Axiom 1. The "only if" part is based on the following Lemma, which also has independent interest.

Lemma 1 *An index of rankiness represents a relation $\mathfrak{M}$ satisfying Axiom 4 if and only if it is a decreasing function of the sum of the ranks of the selected elements.*

Proof. Note first that if r is the sum of the ranks of the selected elements of $(\mathcal{N},\mathcal{K})$, then the sum of the ranks of the selected elements of $(\mathcal{N},\mathcal{N}\backslash\mathcal{K})$ is $\frac{N(N+1)}{2}-r$. When a new element x is added to the set $\mathcal{N}$ and is selected, the new sum of the ranks of the selected elements is $r+\rho(x)+(K-k)$: the original rank, plus the rank of x, plus the sum of the rank of the selected elements with rank above

x, $k \leqslant K$ say. Conversely, suppose that $(\mathcal{N} \cup \{x\}, \mathcal{K} \cup \{x\}) \succsim_{\mathfrak{M}} (\mathcal{N}, \mathcal{K})$. Then it must be the case that $(\mathcal{N} \cup \{x\}, \mathcal{N} \backslash \mathcal{K}) \succsim_{\mathfrak{M}} (\mathcal{N}, \mathcal{N} \backslash \mathcal{K})$. The sum of the ranks of the new mirror image set is $\frac{N(N+1)}{2} - r + (N - \rho(x) - K - k)$: the original rank, plus the rank of the selected elements in the mirror image of $(\mathcal{N} \cup \{x\}, \mathcal{K} \cup \{x\})$ which have rank worse than $\rho(x)$: there are $N - \rho(x)$ such elements and $(K - k)$ are selected in $(\mathcal{N}, \mathcal{K})$. ■

This Lemma implies that a change that makes a selected set more rank-based would make the mirror image of the selected set less rank-based, as required by Axiom 4.

We can now prove the main theorem, by showing that every index of rankiness representing a relation $\mathfrak{M}$ satisfying Axioms 2 and 3 which is a decreasing function of the sum of the ranks of the selected elements is a strictly monotonic transformation of $M(m(\mathcal{N}, \mathcal{K}))$.

Proof of Theorem 1. Given the ordinal property of the ranking determined by the index, any strictly decreasing function of the sum of ranks can be mapped through a monotonic transformation into a decreasing *linear* function of the sum of ranks. That is, any index of rankiness which is a function of the sum of the ranks, r, can be transformed into one that is written as

$$M(m(\mathcal{N}, \mathcal{K})) = a_{K,N} - b_{K,N} r.$$

Moreover Proposition 1 constrains all PERFECT selections have the same value, and all ANTIPERFECT selections also to have the same value: these values can be normalised to 1 and -1 respectively . This implies that for every $N > 1$ and every $K < N$, the following must hold

$$a_{K,N} - b_{K,N} \frac{K(K+1)}{2} = 1, \tag{7}$$

$$a_{K,N} - b_{K,N} \frac{K(2N - K + 1)}{2} = -1. \tag{8}$$

The first condition requires all PERFECT selections to give value 1 to the index and the second all ANTIPERFECT selections to give value -1. Solving the above in $a_{K,N}$ and $b_{K,N}$, we get:

$$a_{K,N} = \frac{N+1}{N-K}, \tag{9}$$

$$b_{K,N} = \frac{2}{K\,(N-K)}. \tag{10}$$

Next, we proceed by induction on N. When $N = 2$, its lowest possible value, there are two possible selections from the set $\{1,2\}$, $K = \{1\}$ which has sum of ranks $r = 1$ and $K = \{2\}$ which has sum of ranks $r = 2$. The former is PERFECT, and so we must have $a_{1,2} - b_{1,2} = 1$, and the latter is ANTIPERFECT, and so $a_{1,2} - 2b_{1,2} = -1$. (9) and (10) satisfy these constraints. In addition, (9) and (10) must also satisfy Axioms 2 and 3.

- When the selection is $\mathcal{K} = \{1\}$, adding a new element to $\mathcal{N}$ can result in a selected set with any of the following images: $\{1,2\}$ (Axiom 2.i or Axiom 3.i), $\{2\}$ (Axiom 2.ii), $\{1\}$ (Axiom 3.i) and $\{1,3\}$ (Axiom 3.ii). Of these $\{1,2\}$ and $\{1\}$ are PERFECT, and so it must be $a_{2,3} - 3b_{2,3} = 1$ and $a_{1,3} - b_{1,3} = 1$: both of these hold. Conversely, $\{2\}$ and $\{1,3\}$ are not PERFECT, and so $a_{1,3} - 2b_{1,3} \in (-1,1)$ and $a_{2,3} - 4b_{2,3} \in (-1,1)$ which hold.

- Similarly, when the selection is $\mathcal{K} = \{2\}$, the add operation can result in any of the following new selected sets: $\{1,3\}$ (Axiom 2.i) $\{2,3\}$ (again Axiom 2.i and also Axiom 3.ii), $\{3\}$ (Axiom 2.ii), $\{2\}$ (Axiom 3.i). Selection $\{2,3\}$ is ANTIPERFECT, and so $a_{2,3} - 5b_{2,3}$ must equal -1, and selection is also ANTIPERFECT $\{3\}$, which requires $a_{1,3} - 3b_{1,3} = -1$. The remaining selections are not ANTIPERFECT, and this requires: $a_{2,3} - 4b_{2,3} \in (-1,1)$, and $a_{1,3} - 2b_{1,3} \in (-1,1)$. All these hold when $a_{K,N}$ and $b_{K,N}$ are given by (9) and (10).

This establishes that the coefficients a and b satisfy (7) and (8) and so are given by (9) and (10) when $N = 2$, and establishes the first step of the induction process. For the second step, assume to have shown that the statement holds for $N - 1$, that is:

$$a_{K,N-1} = \frac{N}{N-1-K}, \tag{11}$$
$$b_{K,N-1} = \frac{2}{K(N-1-K)}, \tag{12}$$

for all $K = 1, \ldots, N-2$. Recall that r_K and $r^{N,K}$ are the lowest and highest possible values for the sum of ranks r when the set has size N and the selection has size K. We must have:

$$\begin{aligned} a_{K-1,N-1} - b_{K-1,N-1} r_{K-1} &= a_{K,N} - b_{K,N} r_K \\ &= a_{K,N} - b_{K,N}\left(r_{K-1} + K\right), \end{aligned} \tag{13}$$

because if the selection is PERFECT then including a new element both in $\mathcal{N}$ and in $\mathcal{K}$ increases the lowest possible sum of the ranks by K. And similarly for the highest possible sum of ranks:

$$\begin{aligned} a_{K-1,N-1} - b_{K-1,N-1} r^{N-1,K-1} &= a_{K,N} - b_{K,N} r^{N,K} \\ &= a_{K,N} - b_{K,N}\left(r^{N-1,K-1} + N\right). \end{aligned} \tag{14}$$

By the induction hypothesis, (11)-(12), we have that

$$a_{K-1,N-1} = \frac{N}{N-K},$$
$$b_{K-1,N-1} = \frac{2}{(N-K)(K-1)}.$$

Substitute these into the two above equations, (13) and (14):

$$\frac{N}{N-K} - \frac{2r_{K-1}}{(N-K)(K-1)} = a_{K,N} - b_{K,N}\left(r_{K-1} + K\right),$$

$$\frac{N}{N-K} - \frac{2r^{N-1,K-1}}{(N-K)(K-1)} = a_{K,N} - b_{K,N}\left(r^{N-1,K-1} + N\right);$$

now substitute

$$\begin{aligned} r_{K-1} &= \frac{(K-1)((K-1)+1)}{2} = \frac{K(K-1)}{2}, \\ r^{N-1,K-1} &= \frac{(K-1)(2(N-1)-(K-1)+1)}{2} = \frac{(2N-K)(K-1)}{2}, \end{aligned}$$

to get:

$$\begin{aligned} \frac{N}{N-K} - \frac{K(K-1)}{(N-K)(K-1)} &= a_{K,N} - b_{K,N}\left(\frac{K(K-1)}{2} + K\right), \\ \frac{N}{N-K} - \frac{(2N-K)(K-1)}{(N-K)(K-1)} &= a_{K,N} - b_{K,N}\left(\frac{(2N-K)(K-1)}{2} + N\right). \end{aligned}$$

Finally, solve the above in $a_{K,N}$ and $b_{K,N}$, to obtain (9) and (10). This establishes the Theorerm. ■

This is the main result of the paper: only the rankiness index (2), or a monotonic transformation of it, can rank selections from sets in a way that respects dominance and mirror invariance. In other words, an index of rankiness which satisfies the dominance Axioms 2-3, and the mirror invariance Axiom 4, is uniquely, up to monotonic transformations, given by (2): Axioms 2, 3 and 4 together characterise the index of rankiness (2).

While any strictly monotonic transformation of the index (2) would represent a relation satisfying the Axioms 2, 3 and 4, the functional form given in (2) has the twin advantages of being linear in the sum of ranks, and of taking value -1 if and only if the selection from the set $(\mathcal{N}, \mathcal{K})$ is ANTIPERFECT, and value 1 if and only if the selection from the set $(\mathcal{N}, \mathcal{K})$ is PERFECT, which is a natural normalisation. It also takes expected value 0 if the selection is completely random: to see this, note that the expected rank of one random draw is $\frac{N+1}{2}$, and so for K random draws is $\frac{K(N+1)}{2}$. Substitute this in (2), to obtain 0.

The structure of the proof of Theorem 1 allows to replace the mirror invariance Axiom, with an equivalent one, which given a different interpretation to the restriction imposed on the rankiness relation to obtain characterisation.

Axiom 5 (Position Irrelevance) *For every* $(\mathcal{N}, \mathcal{K}) \in \mathscr{S}$ *and for every* $y_1 \in \mathcal{N} \backslash \mathcal{K}$ *and* $x_1 \in \mathcal{K}$*, satisfying* $\rho(x_1) = \rho(y_1) + a$*, and any* $y_2 \in \mathcal{N} \backslash \mathcal{K} \cup \{x_1\}$ *and* $x_2 \in \mathcal{K} \cup \{y_1\}$*, satisfying* $\rho(x_2) = \rho(y_2) - a$*, where* a *is any integer such that* $\rho(y_1) + a, \rho(y_2) - a \in \{1, \ldots, N\}$*, then* $(\mathcal{N}, \mathcal{K}) \sim_{\mathfrak{M}} (\mathcal{N}, \mathcal{K} \cup \{y_1, y_2\} \setminus \{x_1, x_2\})$*.*

In words, Axiom 5 considers two subsequent "swaps" between a selected and a non-selected element. The element newly included in the selection and the one removed differ in rank by a. The second swap turns the intermediate selection obtained with the first swap into the final one, by selecting an element not selected and removing another element from the (intermediate) selection which are $-a$ ranks apart. Thus, for example, if the first swap selects the (originally non-selected) 12-th ranked element and de-selects the 19-th ranked; their difference in rank is -7. Suppose the second swap selects the (originally non-selected) 56-th ranked element and de-selects the 49-th ranked; their difference in rank is 7.[12] Then, by Axiom 5, the initial and the final selection are equally rank-based. In shorter, looser words, the effect of a change in the selection depends only on the extent of the change, not on whether it affects the best or the worst elements of the set.[13] We note that

[12]The element swapped in the second swap need not be different from the elements swapped in the first.

[13]A simple example may illustrate this idea: in some sports, a team's success is deter-

the relative importance of the position in the ranking is at the core of the analysis of distance between preferences in Can (2014). It is straightforward to establish the following.

Corollary 2 *A reflexive transitive, complete and continuous rankiness relation* $\mathfrak{M}$ *satisfies Axioms 2, 3 and 5 if and only if it can be represented by a monotonic transformation of the index of rankiness* $M(m(\mathcal{N}, \mathcal{K}))$*, given in (2).*

Proof. This is simply a consequence of Lemma 1 together with the obvious observation that the sum of the ranks is left unchanged by a swap that satisfies Axiom 5. ■

Note that Theorem 1 and Corollary 2 together imply that Axioms 4, mirror invariance, and Axiom 5, position irrelevance, are equivalent.

4 The index M and the Kendall-Tau distance

The study of metrics on orders, initiated by Kendall (1938) has developed a measure of distance between two rankings of the elements of a given set $\mathcal{N}$, the Kendall-Tau distance (Kemeny 1959), recently extended to choice functions (Klamler 2008). This is obtained by counting the number of times the two rankings "switch" two elements $x, y \in \mathcal{N}$. That is, if ρ_1 and ρ_2 are the two rankings, the distance between them is the number of pairs $(x, y) \in \mathcal{N} \times \mathcal{N}$ such that $\rho_1(x) > \rho_2(x)$ and $\rho_1(y) < \rho_2(y)$. This count can then be normalised by the maximum possible number of switches.

mined by the performance of its best athletes. This is typically the case, for example, in "Grand Tour" cycling, where a team's objective is for the team leader to win the race. In rowing (and in team pursuit cycling), on the other hand, everyone must push at the same rate, and the team's result is heavily influenced by the performance of its weakest members. Ranking of selections would, in these sports, violate Axiom 5. Conversely, relays in track and field (especially the 4×400) approximately satisfy Axiom 5.

This idea has been applied to voting mechanism, whereby a ranking is interpreted as a vote, and the outcome of an election as the aggregation of the rankings of different voters (for example, Davenport and Kalagnanam 2004 or Betzler and Dorn 2010). We view the Kendall-Tau distance from different angle: note in the first place that the index of rankiness (2) applies to a broader range of situations, such as those where the selections being compared are from different sets. Secondly, the index (2) compares selections, rather than rankings, as the Kendall-Tau distance does. Nevertheless, a selection does rank the elements of a set, albeit in a very coarse manner: the selected elements are joint first, and the non-selected ones are joint $(K+1)$-th. Recall that the Kendall-Tau distance is defined when there are ties, and so we can measure the Kendall-Tau distance between the ranking determined by a selection and a given ranking ρ of $\mathcal{N}$. In this section we study the relation between this Kendall-Tau distance and the index of rankiness (2), $M(\mathcal{N},\mathcal{K})$.

To develop the formal analysis, given a selected set $(\mathcal{N},\mathcal{K})$, define a mapping $\kappa^{\mathcal{N}\mathcal{K}} : \mathcal{N} \longrightarrow \{1,\ldots,N\}$ as follows: [14]

$$\kappa^{\mathcal{N}\mathcal{K}} : x \longmapsto \begin{cases} 1 & \text{if} \quad x \in \mathcal{K}, \\ K+1 & \text{if} \quad x \in \mathcal{N}\backslash\mathcal{K}. \end{cases}$$

We can label $\kappa^{\mathcal{N}\mathcal{K}}$ the ranking induced by the selection $\mathcal{K}$. Since it is a ranking of the set $\mathcal{N}$, we can define the Kendall-Tau distance between ρ and $\kappa^{\mathcal{N}\mathcal{K}}$. To count the number of "switches", between rankings ρ and $\kappa^{\mathcal{N}\mathcal{K}}$, note that a "switch" occurs only when the better element is not selected and the

[14]That is, $\kappa^{\mathcal{N}\mathcal{K}}$ is the PERFECT selection from a set $\mathcal{N}$ ranked by a relation R such that xRy if and only if $\rho(x) < \rho(y)$ Note that, while a given relation R determines a PERFECT selection uniquely, a given PERFECT selection can be the result of several different relations: to be precise, there are $K!\,(N-K)!$ different relation-determined rankings such that, if the best K elements are selected, determine the same selection $\mathcal{K}$ from a given set $\mathcal{N}$.

worse one is. The total number of switches is thus measured by:

$$\frac{\left|\kappa^{\mathcal{NK}}(x)-\kappa^{\mathcal{NK}}(y)\right|}{2K}\left(1-\frac{\rho(x)-\rho(y)}{|\rho(x)-\rho(y)|}\frac{\kappa^{\mathcal{NK}}(x)-\kappa^{\mathcal{NK}}(y)-\varepsilon}{|\kappa^{\mathcal{NK}}(x)-\kappa^{\mathcal{NK}}(y)-\varepsilon|}\right),$$

where $\varepsilon \in (0,1)$, and where the subscript is omitted from κ. To see this, note simply that the first term is $\frac{1}{2}$ if only one of x and y is selected, and is 0 otherwise. Consider the second factor: given that $\kappa^{\mathcal{NK}}(x) \neq \kappa^{\mathcal{NK}}(y)$, its second term is 1 if ρ and κ agree, and -1 if they do not, and so the whole term is 2 if a switch occurs. The only role of ε is to ensure that the denominator is not 0. To sum up, the Kendall distance, between the ranking of a set $\mathcal{N}$ and the selection $\mathcal{K}$ from it given by the total number of switches, is given by

$$\tau(\mathcal{N},\mathcal{K})=\sum_{x\in\mathcal{N}}\sum_{y\in\mathcal{N}\setminus\{x\}}\frac{\left|\kappa^{\mathcal{NK}}(x)-\kappa^{\mathcal{NK}}(y)\right|}{2K}\left(1-\frac{\rho(x)-\rho(y)}{|\rho(x)-\rho(y)|}\frac{\kappa^{\mathcal{NK}}(x)-\kappa^{\mathcal{NK}}(y)-\frac{1}{2}}{\left|\kappa^{\mathcal{NK}}(x)-\kappa^{\mathcal{NK}}(y)-\frac{1}{2}\right|}\right).$$

Next note that the maximum number of switches is $(N-K)\,K$, which happens in an ANTIPERFECT selection, where the first $N-K$ elements have K switches each, and therefore the normalised Kendall-Tau distance is

$$\hat{\tau}(\mathcal{N},\mathcal{K})=\frac{\tau(\mathcal{N},\mathcal{K})}{K\,(N-K)}. \tag{15}$$

One would want that the shorter the distance between the given ranking in the set $\mathcal{N}$ and the ranking induced by the selection $\mathcal{K}$, the more rank-based the selected set $(\mathcal{N},\mathcal{K})$. The next results shows that this is indeed the case: it establishes the equivalence between the normalised Kendall-Tau (15) distance and the index of rankiness (2).

Proposition 4 $M(m\,(\mathcal{N},\mathcal{K}))=1-2\hat{\tau}(\mathcal{N},\mathcal{K})$.

Proof. We proceed by induction on N, the number of elements of N. It is trivially true for $N = 2$. In this case, $K = 1$: when the better (worse) element is selected $M(m(\mathcal{N}, \mathcal{K})) = 1$ and $\hat{\tau}(\mathcal{N}, \mathcal{K}) = \tau(\mathcal{N}, \mathcal{K}) = 0$ ($M(m(\mathcal{N}, \mathcal{K})) = -1$ and $\hat{\tau}(\mathcal{N}, \mathcal{K}) = \tau(\mathcal{N}, \mathcal{K}) = 1$). Next, suppose to have demonstrated the result for $N - 1$; let K be the number of selected elements, so we have

$$M(m(\mathcal{N}, \mathcal{K})) = 1 - 2\hat{\tau}(\mathcal{N}, \mathcal{K}),$$

and

$$M(m(\mathcal{N}, \mathcal{K})) = \frac{N - \frac{2}{K}\sum_{x \in \mathcal{K}} \rho(x)}{N - 1 - K} = \frac{K(N - 1 - K) - 2\tau(\mathcal{N}, \mathcal{K})}{K(N - 1 - K)}. \tag{16}$$

which implies:

$$\sum_{x \in \mathcal{K}} \rho(x) - \tau(\mathcal{N}, \mathcal{K}) = \frac{1}{2} K(K + 1) \tag{17}$$

Now increase the number of elements in the set from $N-1$ to N, which is achieved via the inclusion in $\mathcal{N}$ of a new element, z. Its rank in the new set $N \cup \{z\}$ is $\rho(z) \in \{1, \ldots, N\}$, and z is either selected, giving the new selected set $(\mathcal{N} \cup \{z\}, \mathcal{K} \cup \{z\})$, or not selected, giving the set $(\mathcal{N} \cup \{z\}, \mathcal{K})$. Consider the first case: the new number of selected elements is $K + 1$. In this case note that the total rank of the selected elements increases by $\rho(z) + s$, where s is the number of elements in the set $\{y \in \mathcal{K} | \rho(y) > \rho(z)\}$. The new value of τ, is instead equal to the previous one, $\tau(\mathcal{N}, \mathcal{K})$, increased by the number of new switches generated by z: all the previous switches remain such of course. This is given simply by the number of elements in the set $\{y \in \mathcal{N} \backslash \mathcal{K} | \rho(y) < \rho(z)\}$, which can be obtained by noting that there are $\rho(z) - 1$ elements ranked better than z, and that $K - s$ are selected (because s is the number of selected elements with rank worse than z). Thus the increase in τ is $(\rho(z) - 1 - (K - s))$. This gives:

$$\begin{aligned} M(m(\mathcal{N} \cup \{z\}, \mathcal{K} \cup \{z\})) &= \frac{N + 1 - \frac{2}{K+1}\sum_{x \in \mathcal{K}} \rho(x) - (\rho(z) + s)}{(K + 1)(N - K - 1)} \\ &= \frac{(K + 1)(N - K - 1) - 2\tau(\mathcal{N}, \mathcal{K}) - 2(\rho(z) - 1 - (K - s))}{(K + 1)(N - K - 1)}. \end{aligned}$$

Rearrange to obtain (17) again. Suppose instead z is not selected, and the new set is $(\mathcal{N} \cup \{z\}, \mathcal{K})$. The total rank of the selected elements increases only by s, where again s is the number of elements in the set $\{y \in \mathcal{K} | \rho(y) > \rho(z)\}$. The new value of τ is instead equal to the previous one, $\tau(\mathcal{N}, \mathcal{K})$, increased by the number of new switches generated by z, which in this case is s, the number of selected elements with rank worse than z. Hence we can write

$$M\left(m\left(\mathcal{N} \cup \{z\}, \mathcal{K}\right)\right) = \frac{N + 1 - \frac{2}{K}\left(\sum_{x \in \mathcal{K}} \rho(x) - s\right)}{N - K} = \frac{K(N-K) - 2\tau(\mathcal{N}, \mathcal{K}) - 2s}{K(N-K)}.$$

which again gives (17) and completes the proof. ■

Thus, in addition to satisfying natural axioms, the index (2) coincides, in the situations where both can be applied, with an established measure of distance between rankings. This paper therefore provides a micro-foundation of the Kendall-Tau distance, which currently lacks one.

5 Example: rankiness in Italian universities

The rankiness index proposed here finds a natural application in the analysis of promotions in hierarchical organisations, where, at given intervals, individuals from the pool of potential candidates are assessed and some are promoted, some are not. The determinants of promotions may be stated formally or known implicitly: thus for example, academic promotions may be decided by criteria ranging from scientific productivity, to teaching performance, fund-rising ability, seniority, or age; the relative importance of performance along the various criteria may of course vary from institution to institution and from discipline to discipline.

In the Italian academic sector, the separation between environments is very formal and rigid, and so it is relatively simple to compare them, and in

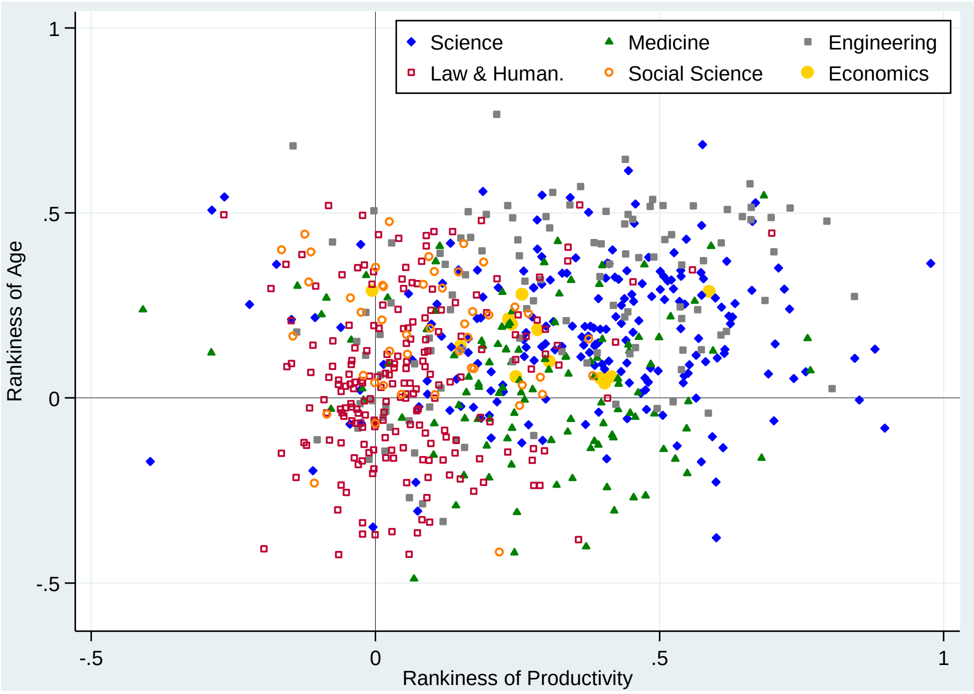

Figure 1: Kernel density of the index of rankiness by scientific sector.

this section we sketch how our rankiness index can describe the promotion process in Italian universities.

As explained in Checchi et al (2014), academic careers in Italy are firmly channelled along narrowly defined research fields: every academic is allocated to one and only one of 371 scientific sectors (SSDs), changing sector is relatively unusual, and the members of the promotion panels in each scientific sector are chosen exclusively among academic post holders in that sector.

The dataset assembled by Checchi et al (2014) allows us to rank all the candidates for promotion to associate professor in each scientific sector in the period from 1995 to 2011 according to two criteria, their record of publication in international journals[15] and their age.

[15]Details are again in Checchi et al (2014): for each candidate we construct a score given by a combination of research output and impact: the former measured by the number of

It is then a simple matter to calculate the rankiness according to the two rankings of the selections made by the panels in each scientific sector, in each of the four separate sub-periods which Checchi et al (2014) suggest to aggregate calendar years to reflect the pattern of the promotion rounds.

Figure 1 describes this construction. Each dot corresponds to one of 371 scientific sectors, except the smaller ones, in one 4-year interval. The abscissa of a dot is the rankiness according to scientific productivity, its ordinate the rankiness according to age.[16] Broad scientific areas are colour coded; we have also singled out economics and econometrics among the social science sectors. An analysis of the Italian university sector might use these indices as characteristics of the selection procedures used in promotion and appointments. A preliminary visual analysis suggests that overall productivity matters more than age, and that it matters more in STEM subjects.

6 Concluding remarks

Often, an agent chooses a number of options from a larger set, the elements of which can be ranked in some objective or generally accepted manner. We propose a way to assess how close a selection is to the ranking. Aside from its intrinsic interest in the examples given in the introduction, availability of this measure might address the need reported in the medical literature for an objective evaluation of clinical services (Iverson 1998, Bickman 2012), or help the study of aspects of social mobility, such as the importance of a

publication listed in the Thomson Reuters Web of Knowledge dataset, and the latter by the individual h-index.

[16]Occasionally, the rankings we constructed in this way display ties. The analysis of this paper applies to antisymmetric relations on the set $\mathcal{N}$, and thus it excludes ties at the outset. To break the ties in the construction of the rankiness index displayed in Figure 1, we have followed a randomisation approach, by bootstrapping the rankiness index (2) over many repetitions of the procedure, whenever a scientific sector's ranking has ties.

person's family position in her access to leadership positions in society. In this paper, we require that the comparison between selections satisfies some natural dominance requirements (Axioms 2 and 3), and the requirement that the mirror image of a change that nears a selection to the ranking must push the mirror image of that selection away from the ranking (Axiom 4). These three simple axioms prove very strong, in that they identify a unique index which unambiguously ranks any selection from any set. This index has a very simple expression, which depends only on the sum of the ranks of the selected elements, and the number of elements in the set and in the selection. The paper ends with a specific example of the potential applications of our index to the Italian university system.

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