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A new index combining the absolute and relative aspects of income poverty: Theory and application[a]

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Abstract

I derive a new index combining the absolute and relative aspects of income poverty. Earning a larger income decreases one's absolute poverty but experiencing a larger income inequality increases one's relative poverty. Provided that the individual poverty is not computed based on the normalized income, the two aspects can be weighed such that absolutely poor individuals are always considered poorer than relatively poor individuals. Only the value of poverty aversion associated with the Poverty Gap Ratio is consistent with this approach. An application illustrates that the new index yields intuitive judgments about unequal growth experiences, for which all absolute (resp. relative) poverty indices systematically conclude that poverty has decreased (resp. increased).

Keywords: Income Poverty Measure, Relative Poverty, Absolute Poverty, Income Inequality, Poverty Lines, Decomposable Index.

1   Introduction

Income poverty reduction is a major political objective, both at national and international levels. In the past decade, policy makers such as the EU Commission or the World Bank have adopted quantified poverty reduction targets.[1] These targets are based on income poverty measures, which are composed of two elements: a poverty line and an index (Sen, 1976). A poverty line specifies the income threshold below which individuals are considered to be poor. An index aggregates the poverty of all individuals in a society and, hence, allows us to compare poverty in different societies.

There exist two central approaches for measuring income poverty, absolute poverty and relative poverty. They differ in the type of poverty line used. An absolute line has its income threshold independent of the standard of living whereas a relative line's income threshold evolves as a constant fraction of the standard of living. These two types of lines aim at capturing different deprivations. On the one hand, absolute poverty refers to the idea of *subsistence*. An individual is absolutely poor if her income is not sufficient to satisfy several of her basic needs, such as being sufficiently nourished. In a first approximation, the real cost of subsistence is absolute as it does not depend on standards of living. For example, 100 grams of rice contain the same amount of calories in New-York or in New-Delhi. On the other hand, relative poverty refers to the ideas of *social participation* or *inclusion*. An individual is relatively poor if her income is not sufficient to engage in the everyday life of her society (Townsend, 1979; Sen, 1983). The real cost of not being excluded from social participation is relative as it depends on standards of living. The archetypical example is that of the linen shirt (Smith, 1776). Adam Smith observed that in the England of his time people would be too ashamed to appear in public without wearing a linen shirt, which he argued was not the case in the Roman Empire that had a lower standard of living.[2]

Many policy makers aim at reducing both the absolute and relative poverties. These two objectives appear for example in the poverty reduction target of the EU Commission or in the new twin goals of the World Bank.[3] Against absolute poverty policy makers implement pro-growth policies, which typically reward efforts at the potential cost of increasing inequalities. Increasing the income of a poor individual improves her absolute poverty but increasing the inequality she experiences worsens her relative poverty. Against relative poverty they implement redistributive policies, which may distort incentives. Of course, not

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[1]See World Bank (2015) or European Commission (2015).

[2]The normative foundations for taking a relativist approach in poverty measurement are reviewed in Ravallion (2008). For instance, Sen (1983) made the case that an absolute level in the space of capabilities translates into a relative level in the space of resources. Townsend (1979) discussed how individuals not having the resources for obtaining the living conditions that are widely encouraged in their society would be excluded from ordinary living patterns, customs and activities. Runciman (1966) pointed out that the comparison of own income with incomes of better-off individuals creates a feeling of deprivation.

[3]In its EU2020 strategy, the EU Commission targets to reduce by 20 millions the number of individuals that are *at risk of poverty or social exclusion* (AROPE). The AROPE individuals are *inter alia* those individuals that are *at risk of poverty* (relative poverty) or are *severely materially deprived* (absolute poverty). In 2013, the World Bank committed itself to twin goals: eliminating extreme poverty (absolute poverty) and boosting shared prosperity (relative poverty). The second objective has a clear relative flavor since it is defined as raising the living standards of the bottom 40% of individuals *in any given country*.

all policies induce a trade-off between growth and equality. Nevertheless, one policy seldom dominates all the alternative policies in both dimensions.

As the two objectives are not always aligned, policy makers must regularly arbitrate between them. Trading-off absolute and relative poverty amounts to answering the following question: when does *unequal growth* alleviate income poverty? A country experiences unequal growth if its economic growth goes along with an increase in income inequality. That is, all individuals get more resources but the additional resources go disproportionately more to the middle class and the rich than to the poor.

One serious difficulty is that the two measurement approaches make opposite extreme judgments on unequal growth. Hence, they evaluate very differently the merits of development programs leading to unequal growth. On the one hand, absolute measures evaluate growth positively, regardless of its distribution. On the other hand, relative measures judge positively any reduction in the inequality experienced by the poor, regardless of the poor's income level. Clearly, neither absolute measures nor relative measures are able to make this trade-off. Measuring both forms of poverty in parallel does not solve the issue since, more often than not, the two approaches yield opposite conclusions.[4]

This paper proposes a new way to measure poverty that combines the absolute and relative aspects of income poverty. Previous attempts to develop such a measure followed two different routes. One route measures both forms of poverty in parallel before looking for a way to aggregate them (Atkinson and Bourguignon, 2001; Anderson and Esposito, 2013). Unfortunately, this approach is confronted to several difficulties, including double counting issues. The other route aims at developing a single measure based on a poverty line making the trade-off between the absolute and relative aspects of income. So far, this second route has mostly focused on defining new poverty lines. The most influential proposals of such *endogenous* lines are the hybrid lines (Foster, 1998) and the weakly relative lines (Ravallion and Chen, 2011). Surprisingly, the indices to use in combination with an endogenous line have not been rigorously studied. In empirical applications (Chen and Ravallion, 2013), the default practice is to use an endogenous line in combination with an index derived for absolute lines, such as the popular Foster-Greer-Thorbecke (FGT) indices (Foster et al., 1984).

As shown in the paper, there is a serious limitation associated with measures composed of an endogenous line and an FGT index. These endogenous measures weigh the absolute and relative aspects of income poverty in a questionable way. They may consider that absolutely poor individuals in low-income countries are *less poor* than relatively poor individuals in middle- and high-income countries. The problem is so serious that these endogenous measures may conclude that there is more poverty in the latter countries than in low-income countries. In the application, these measures deem Brazil equally or more poor than Ivory Coast in 2010. Even if income inequality was larger in Brazil than in Ivory Coast, such judgment could be seriously questioned given that mean income in Brazil was more than four times larger than that of Ivory Coast. Moreover, 22.7 % of individuals in Ivory Coast lived on less than $1.25 a day – the World Bank's threshold for extreme poverty (Ravallion et al., 2009) – but only 5.4% in Brazil.

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[4]A common practice is to use absolute measures in low- and middle-income countries and relative measures in high-income countries. Official national poverty definitions mostly follow this practice (Ravallion, 2012) that leads to extreme judgments as explained above.

Why do measures combining an endogenous line with an FGT index yield this debatable conclusion? FGT indices implicitly attribute to each individual a value of *individual poverty* that depends only on her *normalized income*, i.e. her income divided by the income threshold in her society. In 2010, an individual living on \$1 a day in Ivory Coast has the same normalized income as an individual living on \$3.6 a day in Brazil for the weakly relative line used by Chen and Ravallion (2013). As a result, FGT indices attribute to both the same individual poverty. This conclusion ignores that, unlike the latter, the individual in Ivory Coast is below the threshold for extreme poverty. Being extremely poor is not reflected in normalized incomes. Hence, an extremely poor individual in Ivory Coast can be deemed less poor than a non-extremely poor individual in Brazil. This problem is not limited to indices based on normalized incomes but is rather pervasive. It also affects indices based on absolute gaps, i.e. the distance between the threshold and the individual income.

This paper proposes a new index combining the absolute and relative aspects of income poverty. In order to avoid the problem faced by standard indices, I depart from individual poverty comparisons based on normalized incomes or on absolute gaps. To begin with, I define an absolute poverty threshold, which in the application is fixed at \$1.25 a day. Below this absolute threshold, an individual is deemed absolutely poor and her individual poverty does not depend on the standard of living in her society. For instance, two individuals living with \$1.25 a day in Ivory Coast and Brazil contribute identically to poverty in their respective countries. Then, I define the endogenous poverty line above the absolute threshold. An individual above the absolute threshold but below the endogenous line is deemed relatively poor. Her individual poverty depends on the standard of living in her society. In the application, an individual living on \$2 a day in Ivory Coast, where the mean is \$3 a day, contributes identically to poverty as an individual living on \$6.8 a day in Brazil, where the mean is \$13.8 a day. The constraints I impose on individual poverty comparisons imply that absolutely poor individuals are always considered poorer than relatively poor individuals. This judgment is in line with largely shared intuitions, as appeared from questionnaire studies run all over the world (Corazzini et al., 2011).

I define a family of extended FGT poverty indices that meets the new constraints on individual poverty comparisons. For a given value of the absolute threshold this family depends on two parameters, one of which is the poverty aversion parameter. I investigate which members of this new family satisfy compelling properties. The result shows that a unique index satisfies two basic properties. One property is classical and requires that a progressive transfer between two poor individuals does not increase poverty. The other property is new and specific to indices based on endogenous lines. It requires that destroying part of the income of a poor individual does not reduce poverty. This property excludes all values of poverty aversion except the one associated to the Poverty Gap Ratio.

The index characterized is new and inherits the properties of the constraints put on individual poverty comparisons. That is, absolutely poor individuals are distinguished from relatively poor individuals and the former are always considered poorer than the latter. Being additive, the new index is decomposable between the respective contributions of absolutely and relatively poor individuals. This last feature simplifies the analysis of the evolution of poverty and its communication.

Finally, a poverty measure based on the new index is applied to World Bank data. This application illustrates that the judgments obtained from the new measure are more in line with general intuitions than those obtained with standard measures. For instance, the new measure deems Brazil less poor than Ivory Coast. In a second step, the new measure is used to assess the evolution of poverty in several countries that experienced unequal growth. Urban China constitutes a prominent example because it experienced over the period 1990 – 2010 a strong growth together with a sharp increase in inequality. The new measure concludes that poverty in urban China was reduced by about 75% over this period. By decomposing the measure, one can see that this improvement almost entirely rests on the drastic reduction in absolute poverty. Absolute poverty accounted for about two-third of income poverty in 1990, but less than 10% in 2010. This shows that if the main issue in urban China was absolute poverty in 1990, it has become relative poverty in 2010. Studying different countries shows that the measure may yield different judgments on unequal growth. Over the period 1990 – 2010, income poverty did not change in Mexico as the reduction in absolute poverty was compensated by the increase in relative poverty. Over the period 1996 – 2010, unequal growth has lead to an increase in poverty in Hungary where the impact on relative poverty was dominant. In general, whether unequal growth reduces the poverty measure or not depends on the initial importance of absolute poverty.

The paper is organized as follows. The framework, the limitation associated to FGT indices and the family of extended FGT indices are presented in Section 2. The new index proposed is characterized and discussed in Section 3. Other income standards than the mean are discussed in Section 4. The empirical illustration is presented in Section 5. I conclude in Section 6.

2 General Framework

2.1 Notation and definitions

Let an income distribution $y := (y_1, \ldots, y_n)$ be a list of non-negative incomes sorted in non-decreasing order ($y_1 \leq \cdots \leq y_n$). All incomes are expressed in real terms. That is, inflation effects and purchasing power differences have been removed. Mean income $\overline{y} := \frac{\sum y_i}{n}$ is the income standard capturing the standard of living in a distribution. This choice and the robustness of the results for other income standards are discussed in Section 4. Letting $N := \{n \in \mathbb{N} | n \geq 3\}$, the set of income distributions considered is $Y := \{y \in \mathbb{R}_+^N\}$.

An endogenous poverty line is defined by its threshold function $z : \mathbb{R}_+ \to \mathbb{R}_+$, a continuous function specifying the income threshold $z(\overline{y})$ associated to $\overline{y}$. Individual i qualifies as poor if $y_i < z(\overline{y})$. The number of poor individuals is denoted by q. As income distributions are sorted, if $i \leq q$ then individual i is poor.

Endogenous lines generalize many types of poverty lines. An absolute lines is defined by a constant threshold $z^* \in \mathbb{R}_{++}$. A relative line has its threshold evolve as a constant fraction of the income standard, e.g. $z(\overline{y}) = \frac{1}{2}\overline{y}$. Foster (1998) proposes *hybrid* lines, which feature a constant income elasticity $\rho \in [0,1]$.[5]

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[5] For a given income standard, letting z_a be the threshold of an absolute line and z_r be the threshold of a relative line, the hybrid threshold is given by $z_h = z_a^\rho z_r^{1-\rho}$.

This income elasticity can be interpreted as the extent to which poor individuals should share the benefits of economic growth for not becoming poorer. Absolute lines have an income elasticity of zero and relative lines have an income elasticity of one, representing two extreme views on this parameter.[6] A different proposal by Ravallion and Chen (2011) suggests using *weakly relative* lines, whose income elasticity is zero for low-income countries and then increases with standards of living, tending ultimately to a value of one.

I define *monotonic* lines, a subset of endogenous lines that excludes absolute lines but includes relative lines, hybrid lines and weakly relative lines. Let $s(\overline{y})$ denote the slope of line z at mean income $\overline{y}$. Formally, $s(\overline{y})$ is the first order derivative of function z at $\overline{y}$.[7] Restriction 1 excludes strictly decreasing lines $(0 \leq s(\overline{y}))$ and lines exhibiting an excessive sensitivity to the mean $(s(\overline{y}) \leq 1)$.[8] Restriction 2 requires that the slope is strictly positive for at least one value of mean income $\overline{y}^*$ for which poverty-free income distributions exist $(z(\overline{y}^*) \leq \overline{y}^*)$. For non-absolute lines, these two restrictions are arguably very mild. The definition of monotonic lines includes almost all non-decreasing endogenous lines.

Definition 1 (Monotonic line).
An endogenous poverty line z is monotonic if

1. *its slope $s(\overline{y}) \in [0,1]$ for all $\overline{y} \geq 0$, and*

2. *there exists $\overline{y}^* \geq 0$ with $z(\overline{y}^*) \leq \overline{y}^*$ such that $s(\overline{y}^*) > 0$.*

Throughout, I assume that a monotonic line z has been selected. For a given line z, an index ranks all distributions in Y. Let a poverty index be a function $P : Y \rightarrow [0,1]$ representing a complete ranking on Y. For any two $y, y' \in Y$, there is strictly more poverty in y than in y' if $P(y) > P(y')$, and weakly more if $P(y) \geq P(y')$. Observe that y and y' could be distributions associated to two different countries or to the same country at different points in time. The properties of an index P associated with a monotonic line z are investigated in the remainder of the paper.

2.2 Limitation of FGT indices

As shown in the nice review of Zheng (1997), many different poverty indices have been proposed. In spite of this diversity, virtually all empirical applications use poverty indices belonging to the famous FGT family.

$$P^{FGT}(y) \coloneqq \frac{1}{n} \sum_{i=1}^{q} \left(1 - \frac{y_i}{z(\overline{y})} \right)^{\alpha}. \tag{1}$$

The FGT family has a unique parameter $\alpha \in [0, \infty)$, which can be interpreted as poverty aversion. The larger α, the higher is the priority given by the index

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[6]Madden (2000) estimates empirically an upper-bound for the value of this parameter using Irish data.

[7]Function z is assumed continuous but need not be differentiable everywhere. At any $\overline{y}$ at which z is not differentiable, we define the slope as $s(\overline{y}) \coloneqq \lim_{x \to \overline{y}+} \partial z(x)$, where ∂ is the symbol for first order derivative and the symbol $+$ indicates that the limit consider values strictly larger than $\overline{y}$.

[8]If $s(\overline{y}) > 1$, then a non-poor individual whose income is equal to the income threshold could become poor after the distribution of an equal increment to all individuals.

to individuals at the bottom of the income distribution. This family allows for a wide variety of poverty aversion choices and admits the Head-Count Ratio (HC) and the Poverty Gap Ratio (PGR) as particular cases:

$$HC(y) := \frac{q}{n} \qquad \text{for } \alpha = 0,$$

$$PGR(y) := \frac{1}{n} \sum_{i=1}^{q} \left(1 - \frac{y_i}{z(\overline{y})} \right) \qquad \text{for } \alpha = 1.$$

Being additive, FGT indices compute the average of the values of individual poverty in a distribution. The value of individual poverty is zero for non-poor individual. For poor individuals, this value is returned by the function d^{FGT} : $\mathbb{R}_+ \times \mathbb{R}_+ \to [0,1]$, defined as

$$d^{FGT}(y_i, \overline{y}) = \left(1 - \frac{y_i}{z(\overline{y})} \right)^{\alpha}.$$

Any poor individual is attributed by FGT indices a value of individual poverty that depends only on her *normalized income*, i.e. her income divided by the income threshold in her society.

The direct link between individual poverty and normalized income has strong implications. This is true for any type of poverty line. If the line is absolute, then the individual poverty depends only on own income. Therefore, any small increase in income reduces individual poverty, regardless of the progress achieved by the rest of society. If the line is relative, then the individual poverty depends only on the relative income $\frac{y_i}{\overline{y}}$. Therefore, multiplying all incomes in a distribution by a common factor does not affect individual poverties. More generally, if the line is monotonic, then the normalized income does not reflect whether a person is above or below an absolute threshold, such as the threshold for extreme poverty. Therefore, extremely poor individuals in Ivory Coast are attributed the same individual poverty as non-extremely poor individuals in Brazil, where the mean income and the income threshold are larger.

This last point is illustrated by considering incomes in different distributions that are implicitly deemed equivalent by FGT indices. Assume that individual i lives in Ivory Coast and individual j in Brazil. The respective mean incomes in these countries are denoted by $\overline{y}^{Ivo}$ and $\overline{y}^{Bra}$. For FGT indices, i has the same individual poverty as j if

$$\frac{y_i}{z(\overline{y}^{Ivo})} = \frac{y_j}{z(\overline{y}^{Bra})}.$$

For the weakly relative line of Chen and Ravallion (2013), in 2010, an individual living on \$1 a day in Ivory Coast has the same normalized income as an individual living on \$3.6 a day in Brazil. If \$1.25 a day is considered as the threshold for absolute poverty, Figure 1.a shows that some absolutely poor individuals in Ivory Coast are considered less poor than some relatively poor individuals in Brazil. Considering an absolutely poor individual less poor than a relatively poor individual violates largely shared intuitions, as appeared from questionnaire studies run all over the world (Corazzini et al., 2011). As a consequence, FGT indices lead to counter-intuitive poverty conclusions about unequal growth, e.g. consider Brazil to be poorer than Ivory Coast.
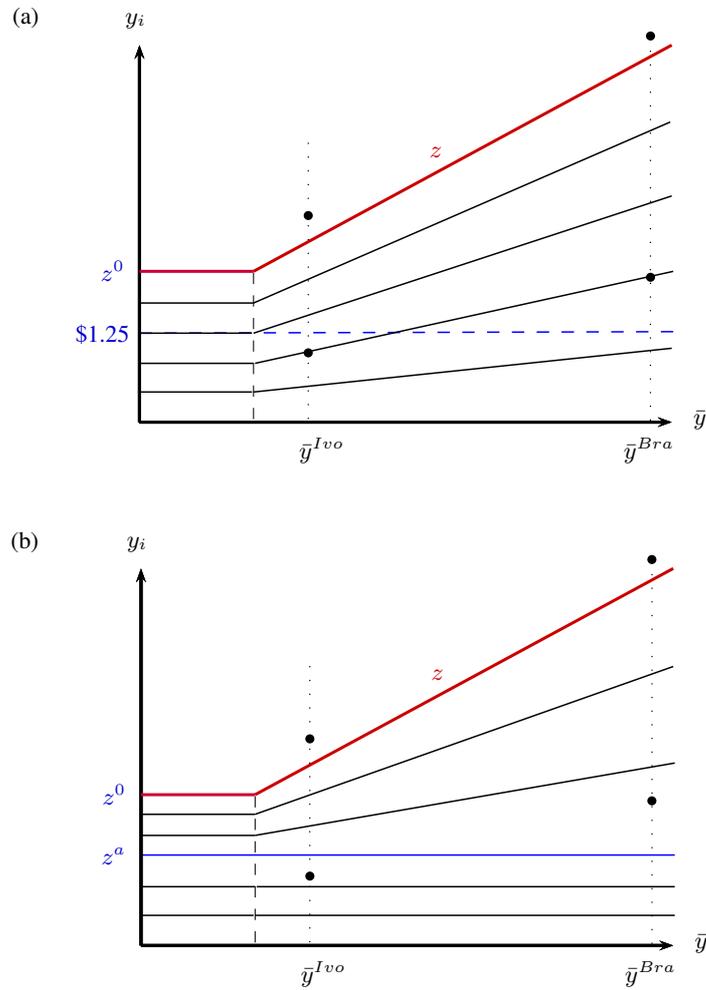
Figure 1: (a) The upper-graph shows, for a given weakly relative line z, the implicit individual poverty comparisons made by FGT indices. Each line below z features constant normalized incomes. The dots mark the different incomes in each distribution. Given the extremely poor individual in Ivory Coast and the relatively poor individual in Brazil are on the same line, they are attributed equal individual poverties. (b) The lines in the lower-graph show the individual poverty comparisons implicit in the definition of the equivalent income function.

This problem is not specific to FGT indices but affect virtually all poverty indices. All indices based on normalized incomes $\frac{y_i}{z(\overline{y})}$ and all indices based on absolute gaps $z(\overline{y}) - y_i$ are affected.[9]

2.3 Fixing FGT indices

I have shown that FGT indices provide very counter-intuitive poverty judgments when associated with a monotonic line. Indeed, FGT indices imply strange comparisons of individual poverty across societies having different standards of living. In this section I propose a new family of indices implying more plausible comparisons of individual poverty.

FGT indices have been widely used partly because of their simple exponential mathematical expression whose parameter is interpretable as poverty aversion. The new family of indices keeps this simple exponential expression but implies different comparisons of individual poverty. Since this family includes FGT indices as particular cases, I call its members *extended FGT indices*.

Before defining the family of extended FGT indices, I present the alternative comparisons of individual poverties on which the family is constructed. These comparisons are encapsulated in the definition of a particular function that I call the *equivalent income function*. This function is defined in two parts separated by a threshold that can be interpreted as the threshold defining absolute poverty.

Let $z^a \in \mathbb{R}_+$ be the threshold defining absolute poverty. This threshold is chosen to be everywhere below the monotonic line z. Denoting by $z^0 := z(0)$ the intercept of the line, the choice of z^a is such that $z^a < z^0$. Any poor individual whose income is strictly above z^a is deemed relatively poor. As $z^a < z^0$, some individuals in low income countries may be deemed relatively poor. That is, the cost of social participation is strictly positive even in low-income societies. Ravallion (2012) defends this point by giving several examples of expenditures playing a social role in low-income countries such as festivals and celebrations.

For a poor individual, the equivalent income at another mean income $\overline{y}^r$ is the amount yielding the same individual poverty as she experiences, but at $\overline{y}^r$. The definition of the equivalent income function is given in equation (2) and illustrated in Figure 1.b. This definition is in two parts. One part specifies the equivalent income for absolutely poor individuals and the other part for relatively poor individuals. Only the individual poverty of relatively poor individuals depends on standards of living. First, two absolutely poor individuals are equally poor when they earn the same income, regardless of the mean incomes in their respective societies. Then, two relatively poor individuals are equally poor when their incomes feature the same normalized distance to z^a. This definition implies that absolutely poor individuals are always considered poorer than relatively poor individuals.

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[9]More generally, all indices satisfying the Scale Invariance property (e.g. FGT indices) and all indices satisfying the Translation Invariance property are affected. See Zheng (1997) for formal definitions.

Definition 2 (Equivalent income function at $\overline{y}^r$).
Take any $\overline{y}^r \geq 0$. For all $y \in Y$ and any poor individual i, the equivalent income function e^r at $\overline{y}^r$ is defined such that:

$$e^r(y_i, \overline{y}) = y_i \qquad if \;\; y_i \leq z^a,$$

$$\frac{e^r(y_i, \overline{y}) - z^a}{z(\overline{y}^r) - z^a} = \frac{y_i - z^a}{z(\overline{y}) - z^a} \qquad otherwise,$$

(2)

where z^a is the threshold for absolute poverty.

I define the new family such that any extended FGT index implies the individual poverty comparisons encapsulated in the equivalent income function for a particular value of z^a. As soon as z^a is non-zero and the line is monotonic, the definition of the equivalent income function makes it not possible for extended FGT indices to have exponential expression at all values of mean income.[10] Therefore, any extended FGT index is based on the additional parameter $\overline{y}^r$, which is the reference value of mean income at which its expression is exponential.

Definition 3 (Extended FGT Family).
Index P belongs to the extended FGT family if there exist $\overline{y}^r \geq 0$ such that

$$P(y) := \frac{1}{n} \sum_{i=1}^{q} \left(1 - \frac{e^r(y_i, \overline{y})}{z(\overline{y}^r)} \right)^{\alpha},$$

(3)

where e^r is the equivalent income function at $\overline{y}^r$.

This family extends the FGT family in the sense that if $z^a = 0$, then this family coincide with the family of FGT indices. The value of individual poverty depends on the normalized equivalent income at $\overline{y}^r$, i.e. the equivalent income at $\overline{y}^r$ divided by the income threshold at $\overline{y}^r$:

$$d(y_i, \overline{y}) = \left(1 - \frac{e^r(y_i, \overline{y})}{z(\overline{y}^r)} \right)^{\alpha}.$$

This family depends on three parameters, namely the poverty aversion α, the reference mean income $\overline{y}^r$ and the absolute threshold z^a. Selecting a particular index requires to fix the values of all three parameters. The selection of an appropriate value for z^a is a normative choice that depends on the type of poverty comparisons that are performed. Yet, given z^a, the values for parameters α and $\overline{y}^r$ must be still be selected. The next section shows that a unique pair of values leads to an index satisfying two very basic properties.

3 The new index

The family of extended FGT indices is very large. Selecting a particular member requires to fix the values for its three parameters. The choice of z^a should

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[10]For FGT indices, the own income and the mean income define the individual poverty in a linear way via the normalized income. The definition of the equivalent income function makes the individual poverty depend non-linearly on the own income and the mean income. Therefore, the mathematical expression of an index depends on the mean income at which it is expressed.

reflect the cost of a bundle of goods allowing to minimally satisfy basic human needs. Therefore, the selection of an appropriate value for z^a can be made relatively easily by the practitioner. Selecting a particular value of poverty aversion is already more difficult, not to mention that of the reference mean income.[11] Fortunately, the values for the latter two parameters can be chosen from two compelling ethical principles. In these sections I present the properties encapsulating these ethical principles and the particular extended FGT index that they jointly characterize.

3.1 Characterization

I require indices in the extended FGT family to respect two properties. For a fixed value of z^a, Theorem 1 shows that a unique index satisfies both properties.

The first property is specific to poverty indices considering both the absolute and relative aspects of income. In such a framework, increasing the income of an individual entails a worse relative situation for the others. Poverty indices must balance those gains and losses without giving excessive importance to relative losses. *Monotonicity in Income* requires that decreasing the income of some *poor* individual never leads to an unambiguous poverty reduction.

Poverty axiom 1 (*Monotonicity in Income*)**.**
For all $y, y' \in Y$, if $y_i < y'_i < z(\overline{y}')$ and $y'_j = y_j$ for all $j \neq i$, then $P(y) \geq P(y')$.

When a poor individual's income increases, her individual poverty decreases as both her absolute and relative situation improve. Yet, the additional income also increases the mean income and this may increase the income threshold. A threshold's increase may have two adverse effects. First, the individual poverties of relatively poor individuals increase.[12] Second, any individual whose income is between the initial and the final values of the income threshold becomes relatively poor. *Monotonicity in Income* requires that the positive impact of such an income increase is dominant. Observe that the larger the number of individuals, the smaller is the impact of a given income increase on the mean income and, hence, on the individual poverties of the others.

It is worth stressing that the Head-Count Ratio, when combined with a monotonic line, may conclude that destroying part of the income of a poor individual reduces poverty. The problem is illustrated in Table 1. The monotonic line z is relative and has its threshold equal to 50% of mean income. The distribution B is obtained from the distribution A by decreasing the income of the poor individual 1. Nevertheless, the HC concludes there is more poverty in distribution A than in B as individual 2 is poor in A but not in B.

Table 1: Index HC violates *Monotonicity in Income* if the line is monotonic.

	y_1	y_2	y_3	$z(\overline{y})$	$HC(y)$
Distribution A	2.5	3	12.9	3.1	$\frac{2}{3}$
Distribution B	2	3	12.9	2.9	$\frac{1}{3}$

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[11]Unlike poverty aversion, the reference value for mean income has no clear normative interpretation.

[12]The individual poverty of absolutely poor individuals is not affected as it is independent on the income standard.

The second property is a standard requirement that most poverty indices characterized in the literature satisfy. *Weak Transfer* requires that a Pigou-Dalton transfer taking place between two poor individuals never unambiguously increases poverty.[13] This property is still compelling when the income standard is the mean income since balanced transfers do not alter the mean. As a result, the individual poverties of individuals not involved in the transfer are preserved.

Poverty axiom 2 (*Weak Transfer*)**.**
For all $y, y' \in Y$ and $\lambda > 0$, if $y_j - \lambda = y'_j > y'_k = y_k + \lambda$, $z(\overline{y}) > y_j$ and $y'_i = y_i$ for all $i \neq j, k$, then $P(y) \geq P(y')$.

I investigate which indices in the extended FGT family respect both properties. It is well-known that poverty indices satisfying *Weak Transfer* are based on convex individual poverties. *Monotonicity in Income* is a new axiom in this context and I show that it has a strong discriminative power.

Assume the value z^a of the absolute threshold has been fixed. In the extended FGT family, each value of poverty aversion defines a subfamily whose members are parameterized by the reference mean income $\overline{y}^r$. For example, the *PGR at $\overline{y}^r$* is the index for which the individual poverty function d is linear in own income at mean income $\overline{y}^r$. The *PGR at the origin* ($\overline{y}^r = 0$) illustrated in Figure 2 plays a key role in the remainder of this paper.

Theorem 1 formalizes the result showing that in the extended FGT family, only the PGR at the origin satisfies *Monotonicity in Income* and *Weak Transfer*.

Theorem 1 (Characterization of the PGR at the origin)**.**
Let P be an index in the extended FGT family based on a monotonic line.

1. *P satisfies* Monotonicity in Income *only if:*

$$\alpha = 1.$$

2. *P satisfies* Monotonicity in Income *and* Weak Transfer *if and only if:*

$$\alpha = 1 \qquad and \qquad \overline{y}^r = 0,$$

   that is, P is the PGR at the origin.

Proof. See the Appendix. ∎

Claim 1 shows that *Monotonicity in Income* is responsible for the largest part of the result. First, among all values of poverty aversion, only the one associated to the PGR is acceptable. This characterization of the poverty aversion's value is due to the exponential mathematical form of the extended FGT family. For the case $\alpha < 1$, when the income of a poor individual tends to the income threshold, the priority granted to her over – say – an absolutely poor individual tends to infinity. Therefore, when the income of an absolutely poor individual increases, the individual poverty of the relatively poor individual close to the income threshold is affected in an adverse way and the index concludes that poverty has increased. The case $\alpha > 1$ is plagued with the reversed problem.

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[13]A Pigou-Dalton transfer is a progressive balanced transfer preserving the relative ranks of the two individuals involved in the transfer.

(a)

$d(y_i, 0)$

1

at $\overline{y} = 0$

$1 - \frac{z^a}{z^0}$

y_i

0   z^a z^0

(b)

$d(y_i, \overline{y}^1)$

1

at $\overline{y} = \overline{y}^1$

$1 - \frac{z^a}{z^0}$

y_i

0   z^a $z(\overline{y}^1)$
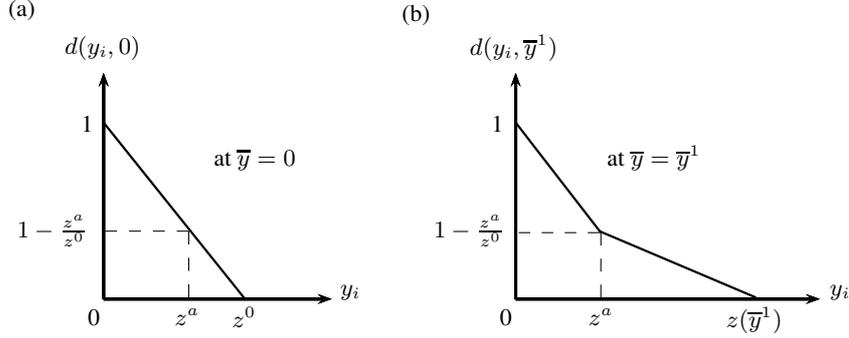
Figure 2: (a) Individual poverty values attributed by the PGR at the origin as a function of y_i for $\overline{y} = \overline{y}^r = 0$. (b) Individual poverty values attributed by the PGR at the origin as a function of y_i for $\overline{y} = \overline{y}^1$ with $z(\overline{y}^1) > z^0$.

When the income of a poor individual tends to the income threshold, her priority over the other poor individuals tends to zero. An increase in her income can be negatively judged by the index. Second, not all members of the PGR at $\overline{y}^r$ subfamily satisfy *Monotonicity in Income*. If the monotonic line is linear, then the PGR at $\overline{y}^r$ satisfies the axiom if and only if $\overline{y}^r$ is below an upper-bound whose value depends on the parameters of the line and the absolute threshold.[14]

Claim 2 shows that *Weak Transfer* further restricts the acceptable members of the PGR subfamily to a unique index. If the reference mean income is not $\overline{y}^r = 0$, then there exist values of mean incomes at which the individual poverty function d is concave, which violates *Weak Transfer*. Here is the intuition for this result. When drawn at the reference mean income, the graph of the individual poverty function d is linear when $\alpha = 1$, as shown in Figure 2.a for the case $\overline{y}^r = 0$. When drawn at a larger mean income than the reference, its graph is piecewise-linear and convex because the income threshold is then larger than at the reference mean income, as shown in Figure 2.b for the case $\overline{y}^r = 0$. If the reference value for mean income is not zero, then there exist values of mean income at which the income threshold is lower than at the reference mean income and the graph is piecewise-linear and concave.

The very sharp conclusions of Theorem 1 are valid for indices in the extended FGT family. The robustness of the result outside this family is investigated in an earlier version of this paper (Decerf, 2015). I show by means of an example that, for other families, the discriminating power of *Monotonicity in Income* is less strong. There is a range of poverty aversion values around the value associated to the PGR for which the index satisfies this property. The PGR at the origin still emerges as the focal index. Any other index satisfying *Monotonicity in Income* and *Weak Transfer* must be close to the PGR at the origin.

If the absolute threshold z^a is non-zero, the PGR at the origin defines a new index of poverty. I present this new index in the coming subsection.

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[14]A linear poverty line has its income threshold defined as $z(\overline{y}) = s\overline{y} + z^0$, where the slope $s \in [0, 1]$. The proof for the existence of this upper-bound is in an earlier version of this paper (Decerf, 2015).

3.2 Presentation

Given an endogenous line z, how can the practitioner compute the index identified above? The first step is to fix the absolute threshold z^a. The value for parameter z^a is selected to be a meaningful threshold for absolute material deprivation for the empirical question tackled by the practitioner.

An important remark relates to the selection of the parameter $z^a \in [0, z^0)$. Given z^0, the larger z^a, the larger is the emphasis placed by P on poverty in low-income countries. Formally, increasing z^a increases the equivalent incomes at origin of relatively poor individuals. Hence, their individual poverties decrease. Crucially, the larger the mean income in a given distribution, the larger is the decrease in individual poverties. For instance, assume that two relatively poor individuals living in countries with different income thresholds have the same individual poverty for some value of z^a. For $z^{a'} > z^a$, the individual living in the country with the larger income threshold is considered less poor than the other individual. In contrast, changing z^a does not alter the equivalent incomes at origin of absolutely poor individuals.

Consider the two extreme values for z^a. If z^a tends to z^0, then the index becomes the PGR based on an absolute line with threshold z^0. In this case, any two income distributions are compared based on the incomes of individuals below z^0. The individual poverties of relatively poor individuals – with income between z^0 and the income threshold – tend to zero. If instead $z^a = 0$, there are no absolutely poor individuals and the index is the classical PGR below the endogenous line, which only depends on normalized incomes. Clearly, the PGR below the endogenous line places more emphasis on poverty in richer countries than the PGR below the absolute line with threshold z^0. These two limit values for z^a are rather extreme and the value of z^a should not be selected near the boundaries of $[0, z^0)$.

Once an appropriate z^a has been chosen, the second step is mechanical and simply amounts to computing the mathematical expression of the index P. This mathematical expression, illustrated in Figure 2, is the PGR at the origin

$$P(y) = \frac{1}{n} \sum_{i=1}^{q} \left(1 - \frac{e^0(y_i, \overline{y})}{z^0} \right),$$

where $e^0(y_i, \overline{y})$ is the equivalent income at $\overline{y}^r = 0$ and z^0 is the intercept of the endogenous line.

The classical PGR can be interpreted as the average percentage shortfall in income, from the income threshold. The individual poverty attributed by the new index is equal to the equivalent percentage shortfall at the origin. Therefore, the new index's interpretation is the average equivalent percentage shortfall at the origin.

Interesting properties of the index

Besides satisfying *Monotonicity in Income* and *Weak Transfer*, the new index has several interesting features.

First, the index makes a clear distinction between absolutely poor individuals and relatively poor individuals. In contrast to standard indices, the latter are never considered to be poorer than the former. This feature follows from the

definition of the equivalent income function. For global poverty measurement, this definition implies that:

- An extra dollar has the same impact on global poverty when it is given to an absolutely poor individual in a low-income country as when it is given to an absolutely poor individual in a middle-income country.

- An extra dollar has more impact on global poverty when it is given to a relatively poor individual in a low-income country than when it is given to a relatively poor individual in a middle-income country. Even if bringing an individual from the subsistence threshold to the poverty threshold has the same impact on her individual poverty in both countries, it is more costly to do so in the middle-income country.

For the evolution of income poverty over time, the definition of e^r implies that:

- Growth, however unequally distributed, decreases the individual poverties of absolutely poor individuals.

- On the contrary, growth should not be too unequally distributed in order for the individual poverties of relatively poor individuals to decrease.

A corollary of the last two bullet points is that this index concludes that growth, if strong enough, eventually eradicates absolute poverty but not necessarily relative poverty. Whether the latter form of poverty eventually disappears depends on the distributive aspects of growth.

Second, being additive, the index is decomposable between the respective contributions of absolutely and relatively poor individuals. Letting q^a denote the number of absolutely poor individuals in the distribution, the index can be decomposed as

$$P(y) = \underbrace{\frac{1}{n}\sum_{i=1}^{q^a}\left(1 - \frac{e^0(\hat{y}_i, \overline{y})}{z^0}\right)}_{P^a(y)} + \underbrace{\frac{1}{n}\sum_{i=q^a+1}^{q}\left(1 - \frac{e^0(\hat{y}_i, \overline{y})}{z^0}\right)}_{P^r(y)}. \tag{4}$$

As shown in the application, the decomposability simplifies the analysis of the evolution of poverty and its communication. Before turning to the application, next section discusses the choice of the mean as the income standard.

4 Income standard

I discuss in this section the choice of the income standard to which the poverty line is sensitive. The choice of income standard is important because it defines the channel through which the other individuals' incomes affect individual poverty. More specifically, it defines the distributional changes altering the income threshold and, hence, the individual poverties. Poverty judgments depend on the income standard used. If the literature on global poverty measurement uses mean-sensitive lines (Atkinson and Bourguignon, 2001; Chen and Ravallion, 2013), median-sensitive lines are often used in practice.[15]

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[15]The AROP measure of the European Commission uses a median-sensitive line. The *At Risk of Poverty* measure is the Head-Count Ratio based on a relative line whose threshold is 60 % of the median income.

In practice, the income standard is not computed from administrative data but often from random samples. The median is known to be more robust than the mean in random samples (Cowell and Victoria-Feser, 1994). Hence, median-sensitive lines have a less volatile income threshold. Therefore, median-sensitive lines are considered superior to mean-sensitive lines when inequality is constant over time.

However, unlike the mean, the median is affected by the inequality in the distribution. de Mesnard (2007) has shown that median-sensitive indices behave very counter-intuitively when income distributions experience an increase in inequality. For instance, policies whose unique impacts are regressive transfers from the middle class to the rich are deemed poverty-reducing. This issue is particularly problematic in a World in which intra-country inequalities are on the rise (Bourguignon, 2013). The evolution of the official poverty measure in New-Zealand over the period 1981 – 1992 constitutes an illustration of the problem. According to Easton (2002), the implementation of policies inducing regressive transfers led to a decrease of the incomes of the bottom 80 % of households. Nevertheless, the median-sensitive HC dropped due to the large decline in median income and some institutions used these figures to argue that the regressive policies were a success. Unsurprisingly, median-sensitive lines yield extreme judgments on unequal growth. Provided that the median income does not change, any unequal growth is deemed poverty reducing.

I have argued that the median income is not a good income standard for judging the evolution of poverty in countries experiencing unequal growth. Rather, other income standards are preferable, such as the mean or a lower partial mean (e.g. mean income among the 99% least rich individuals).[16]

How robust are the results to the choice of the income standards? I study this question in an earlier version of this paper (Decerf, 2015). In a nutshell, the answer goes as follows. Theorem 1 holds if the line is sensitive to a lower partial mean, but not for median-sensitive lines. *Monotonicity in Income* constrains the median-sensitive line rather than the poverty aversion parameter. The reason is that the poor individuals' incomes do not affect the median if the median is above the income threshold. Nevertheless, for any income standard, classical FGT indices imply implausible cross-distributions individual poverty comparisons. The particular equivalent income function proposed solves this issue for any choice of income standard.

5   Empirical illustration

In this section, I apply the new index on World Bank data. First, using different poverty measures, I compare poverty between several low-income low-inequality countries and middle-income high-inequality countries. The judgments obtained by a poverty measure based on the new index are more in line with general intuitions than those obtained by classical FGT measures. Second, I use the poverty measure based on my index in order to evaluate whether the economic growth

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[16]Some argue against the use of mean-sensitive lines that the mean is affected by "outliers". If the sensitivity to outliers is perceived as deeply problematic, then a lower partial mean offers a good compromise. For example, the mean income among the 99% least rich individuals is much less affected by outliers than the mean. The downside of such a partial mean is that regressive redistributions benefiting the 1% richest individuals affect it.
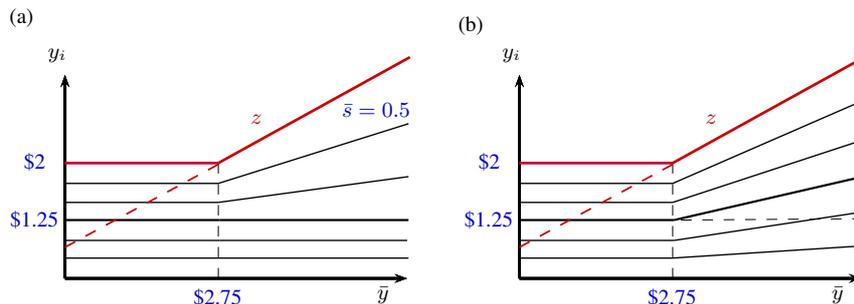
Figure 3: (a) Individual poverty comparisons associated to P^{EL}. The endogenous poverty line is z and the absolute threshold is $z^a = \$1.25$ a day. (b) Individual poverty comparisons associated to PGR^{EL}.

taking place over the last 20 years in several low- and middle-income countries was poverty reducing in spite of the increase in intra-country inequalities. I discuss the variables influencing the answer.

The data is taken from PovcalNet, a website built by the World Bank that provides income and consumption data.[17] This data is gathered from more than 850 surveys of randomly sampled households in 127 low- and middle-income countries between 1981 and 2010. The frequency and precision of the surveys vary from one country to another. In some countries, the surveys focus on income, whereas in others on the value of total consumption. In order to permit cross-country comparisons, the Bank translates the survey data by making use of the Purchasing Power Parity (PPP) exchange rates for household consumption from the 2005 International Comparison Program. The national income distributions presented in PovcalNet are estimated from the survey data. More information about the data can be found in Chen and Ravallion (2013).[18]

5.1 A poverty measure based on the new index

This section demonstrates how to apply the new index. I assume that the selected endogenous line has the following weakly relative definition, illustrated in Figure 3:

$$z(\overline{y}) = max\{\$2, \$0.625 + 0.5\overline{y}\}.$$

Its income threshold is \$2 a day in countries whose mean income is lower than \$2.75 a day. The World Bank considers that \$2 a day is the threshold for income poverty in developing countries. For mean incomes higher than \$2.75 a day, this line has a constant slope of one half. Observe that the intercept \$0.625 of this second part is positive. As a result, the line does not evolve as a constant fraction of the mean.

This line is very close to that used by Chen and Ravallion (2013). The only difference is that the income threshold for low-income countries used by these

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[17]PovcalNet: the on-line tool for poverty measurement developed by the Development Research Group of the World Bank. www.iresearch.worldbank.org/PovcalNet.

[18]PovcalNet is the database used in Chen and Ravallion (2013).

authors is \$1.25 a day, considered by the World Bank as the threshold for extreme poverty.[19] For richer countries, these authors fit their line on national thresholds. Their premise is that thresholds adopted at a country level reflect a balance made between absolute and relative aspects of income. The endogenous line selected is of course debatable but the objective is simply to pick a reasonable line for illustration purposes.

I take z^a to be the threshold for extreme poverty: \$1.25 a day. This threshold was computed as an average of income thresholds in the fifteen poorest countries of the World (Ravallion et al., 2009).[20] Individuals earning less than \$1.25 a day are deemed absolutely poor and those earning more than \$1.25 a day but less than the endogenous threshold are deemed relatively poor. The poverty measure based on the new index is denoted by P^{EL}, where the superscript is meant to indicate that it is based on the endogenous line.

Judgments based on P^{EL} are compared with those obtained by four other measures. Among the four alternative measures, three are based on the Head-Count Ratio while the last is based on the Poverty Gap Ratio. The first measure, HC^{AL}, is the fraction of individuals whose income is below \$1.25 a day. The second, HC^{RL}, is a relative measure corresponding to the fraction of individuals whose income is below the relative line whose threshold is half the mean income. This measure provides some information about the inequality in the distribution. The third measure, HC^{EL}, is the fraction of individuals whose income is below the endogenous line defined above. The last measure, PGR^{EL}, is the classical Poverty Gap Ratio below the endogenous line.

I comment on the relations existing between P^{EL} and PGR^{EL}. For mean incomes below \$2.75 a day, the endogenous line is flat. Therefore, P^{EL} returns equal values as PGR^{EL} for very poor countries. Above \$2.75 a day, P^{EL} systematically returns lower values than PGR^{EL} because the normalized equivalent incomes at origin of relatively poor agents are larger than their normalized incomes. Therefore, if distribution B has a larger mean income than distribution A with $\overline{y}^A = 2.75$ and P^{EL} concludes that there is more poverty in B than in A, then PGR^{EL} draws the same conclusion. Index PGR^{EL} places more emphasis on poverty in richer countries.

5.2 Empirical results

The data extracted from PovcalNet is used for computing the five poverty measures. First, I show that P^{EL} makes poverty judgments on unequal growth that are more in line with general intuitions than those of the other four measures. Remember that for my purpose, the distributions of two countries can equally be interpreted as two distributions corresponding to the same country at different points in time.

Table 2 provides figures for six countries in 2010. The countries are sorted in increasing order of mean income. Three low-income low-inequality countries are considered, namely Ethiopia, Nepal and Ivory Coast. Their mean incomes

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[19] It makes little sense for my purpose to consider that individuals whose income is \$1.25 a day have the same individual poverty than individuals at the poverty line in richer countries since I assume that \$1.25 a day is the threshold for absolute poverty.

[20] Many among these countries establish their national thresholds based on the cost of a bundle of goods whose consumption guarantees to reach a minimal level of physical survival (including a minimal nutrition level). Therefore this choice seems natural for z^a.

amount to $2, $2.2 and $3 a day respectively and their Gini coefficients are 34%, 33% and 43%.[21] Three middle-income high-inequality countries are considered, namely Bolivia, South Africa and Brazil. Their mean incomes amount to $8.3, $8.4 and $13.8 a day respectively and their Gini coefficients are 50%, 63% and 54%.

Table 2: Cross-country comparisons of poverty figures in 2010.

Countries	Mean	Gini	HC^{AL}	HC^{RL}	HC^{EL}	PGR^{EL}	P^{EL}
Ethiopia	2.0	34	30.6	17.7	65.0	23.1	23.1
Nepal	2.2	33	24.8	18.5	56.3	18.7	18.7
Ivory Coast	3.0	43	22.7	30.0	47.6	18.3	17.4
Bolivia	8.3	50	13.4	43.3	48.3	25.3	16.5
South Africa	8.4	63	13.8	57.1	61.3	32.8	17.6
Brazil	13.8	54	5.4	43.1	46.5	22.1	11.7

All poverty measures and the Gini coefficients are expressed in %. Mean incomes are expressed in $ a day (2005 PPP). Source: PovcalNet.

HC^{AL} is strongly negatively correlated with mean income and HC^{RL} is strongly positively correlated with inequality, as measured by the Gini coefficient.[22] HC^{RL} concludes that middle-income countries, having a larger income inequality, have by far the largest poverty. HC^{AL} reaches the opposite conclusion. Hence, it is difficult to balance the absolute and relative aspects of growth on the sole basis of these two measures. The three measures based on the endogenous line are more nuanced. PGR^{EL} places more emphasis on poverty in richer countries and concludes that the two poorest countries are Bolivia and South Africa. In contrast, the two poorest countries according to P^{EL} are low-income countries, namely Ethiopia and Nepal.

Pairwise comparisons illustrate the different judgments made by P^{EL}, PGR^{EL} and HC^{EL}. PGR^{EL} and HC^{EL} conclude that there is less – or approximately equal – poverty in Ivory Coast than in Brazil, even if the fraction of absolutely poor individuals is much higher in the former (22.7 %) than in the latter (5.4%). In contrast, P^{EL} places more emphasis on the absolute aspects of income poverty and concludes that there is more poverty in Ivory Coast than in Brazil. Oppositions of the same type can be found when comparing South Africa with Nepal or Ivory Coast, or when comparing Brazil with Bolivia. Nevertheless, P^{EL} does not always follow the judgments of HC^{AL}. For instance, HC^{AL} concludes that there is much less poverty in South Africa than in Nepal or Ivory Coast. In contrast, acknowledging the very unequal income distribution in South Africa, P^{EL} deems that those three countries have similar levels of poverty.

Table 2 demonstrates that the poverty judgments drawn from P^{EL} can be radically different from those obtained with the other four measures. Moreover, the judgments drawn from P^{EL} seem to be in line with general intuitions. Next,

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[21]The Gini coefficient is a popular measure of inequality. The larger the Gini coefficient, the larger is inequality. The Gini coefficient's values were obtained online from the World Bank Poverty and Equity Database on the 24th of August 2015, www.povertydata.worldbank.org. The Gini coefficient is measured in 2010 for Ethiopia and Nepal; in 2009 for Bolivia, South Africa and Brazil and in 2008 for Ivory Coast.

[22]In the sample, the coefficients of correlations are -0.97 and 0.99 respectively.

P^{EL} is used in order to evaluate the impact of the unequal growth taking place over the period 1990-2010 in different geographic entities.

Table 3 provides the before- and after-growth figures for five geographic entities.[23] All five geographic entities experienced an increase in mean income together with an increase in inequality, as signaled by the evolution of HC^{RL}. P^{EL} allows us to decompose the fraction of poor individuals (HC^{EL}) between those that are absolutely poor (HC^{AL}) and those that are "only" relatively poor. Furthermore, the figure for P^{EL} can be decomposed between the contribution of absolutely poor individuals (P^a) and that of relatively poor individuals (P^r). These decompositions are illustrated in Figure 4 for urban China, the entity that experienced the most drastic evolution over that period.

Table 3: Evaluation of several unequal growth experiences.

Geo Entity	Year	Mean	HC^{RL}	HC^{AL}	HC^{EL}	P^{EL}	P^a/P^{EL}
World	1990	3.0	21.2	43.0	70.7	30.7	0.82
	2010	4.9	26.4	20.8	52.7	17.7	0.66
Urban China	1990	1.9	9.1	23.4	61.2	18.9	0.62
	2010	7.1	21.7	0.6	30.6	4.7	0.08
Costa Rica	1990	7.0	31.5	8.4	40.0	11.4	0.53
	2010	15.3	40.3	2.6	43.7	8.9	0.22
Mexico	1990	7.8	24.1	4.5	29.2	7.4	0.39
	2010	10.6	35.8	0.7	41.2	7.5	0.05
Hungary	1996	8.8	9.8	0.2	16.0	1.7	0.07
	2010	12.5	15.2	0.2	20.1	2.2	0.06

All poverty measures are expressed in %. Mean income is expressed in $ a day (2005 PPP). P^a corresponds to the contribution of absolutely poor individuals to P^{EL}, defined in (4). Source: PovcalNet.

The World and urban China experienced a large decline in income poverty over the period: P^{EL} dropped by 42% and 75%, respectively. In other words, in spite of the increase in income inequality, particularly important in urban China as indicated by HC^{RL}, P^{EL} concludes unambiguously that growth has been poverty reducing.[24] These reductions reflect primarily the changes in absolute poverty. Absolute poverty was a main concern in both entities in 1990, as confirmed by the values in the last column of Table 3. In the World for example, 43% of individuals were absolutely poor in 1990 and these individuals contributed to 82% of P^{EL}. In 2010, only 20.8% of individuals remained absolutely poor in the World, contributing then to 66% of P^{EL}. For urban China, absolute poverty has been almost eradicated over the period. The decrease in P^{EL} in both entities is driven by the large decrease in P^a.

Over the same period, Costa Rica and Mexico experienced a lower reduction in poverty than the World and urban China. P^{EL} dropped by 22% in Costa Rica whereas it returned to its initial value in Mexico. The increase in relative poverty mitigated the significant reduction in absolute poverty achieved by the

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[23]The figures for the World are an aggregate of the figures for the low- and middle-income countries, weighed by their population. The figures for urban China are based on an endogenous threshold computed for the mean income in urban China.

[24]It is the *intra-country* inequality that is accounted for when discussing the evolution of inequality in the World.

two countries. Absolute poverty was an important concern in 1990 – 53% of P^{EL} for Costa Rica and 39% of P^{EL} for Mexico – although not as dominant as for the World and urban China. The fraction of absolutely poor individuals fell from 8.4% to 2.6% in Costa Rica and from 4.5% to 0.7% in Mexico. At the same time, the large increase in inequality in these two countries implied that more individuals were poor in 2010 than in 1990, as shown by HC^{EL}. In Mexico, the large increase in inequality increased significantly P^r and the reduction in P^a only compensated for the increase in P^r.

Hungary experienced a 30% increase of P^{EL} over the period 1996 – 2010, in spite of an increase of 43% of its mean income. Absolute poverty was not an important concern in 1996 – P^a was less than 10 % of P^{EL} in 1996 – and did not change significantly over the period. On the contrary, income inequality increased and 20% of individuals were poor in 2010 whereas only 16 % of individuals were poor in 1996. The increase in P^{EL} is directly driven by the increase in P^r.

Analyzing with P^{EL} several unequal growth experiences has shown that very different conclusions can be drawn by this measure. Different factors influence the conclusions of P^{EL}, such as the extents of growth and increase in inequality. A key factor is the importance for P^{EL} of absolute poverty at the beginning of the period. If absolute poverty is not an important concern, like in Hungary, the increase in inequality entails an increase in P^{EL}.

The distinction between absolutely and relatively poor individuals and the decomposability of the index make it possible to separately track these two forms of poverty and aggregate them in a coherent way. I illustrate this possibility for the case of urban China, shown in Figure 4. In urban China in 1990, 23.4% of individuals were absolutely poor and 37.8% were relatively poor, adding up to 61.2% of poor individuals. Overall, the poverty index for the income distribution in 1990 takes a value of 18.9%. This value of income poverty can be decomposed into the contribution of absolutely poor individuals (11.8%) and that of relatively poor individuals (7.1%). Hence, absolutely poor individuals contributed to 62% of income poverty, which shows that absolute poverty was the main issue in urban China in 1990. In 2010, 0.6% of individuals were absolutely poor and 30% were relatively poor, adding up to 30.6% of poor individuals. Overall, the poverty index for the income distribution in 2010 takes a value of 4.7%, a figure 75% lower than that of 1990. This lower value of income poverty can be decomposed into the contribution of absolutely poor individuals (0.4%) and that of relatively poor individuals (4.3%). Hence, absolutely poor individuals contributed to 8% of income poverty. This demonstrates that the reduction in absolute poverty is responsible for most of this three-quarters reduction in income poverty. Moreover, it shows that relative poverty became the main issue in urban China in 2010.

Altogether, P^{EL} confirms that poverty reduction has been impressive over the last decades in low- and middle-income countries ("the World" in Table 3). In fact, poverty decreased even more than Head-Count based measures suggest. Over the period 1990-2010, even if the fraction of poor individuals decreased only by 25% , P^{EL} concludes that income poverty was reduced by 42%.
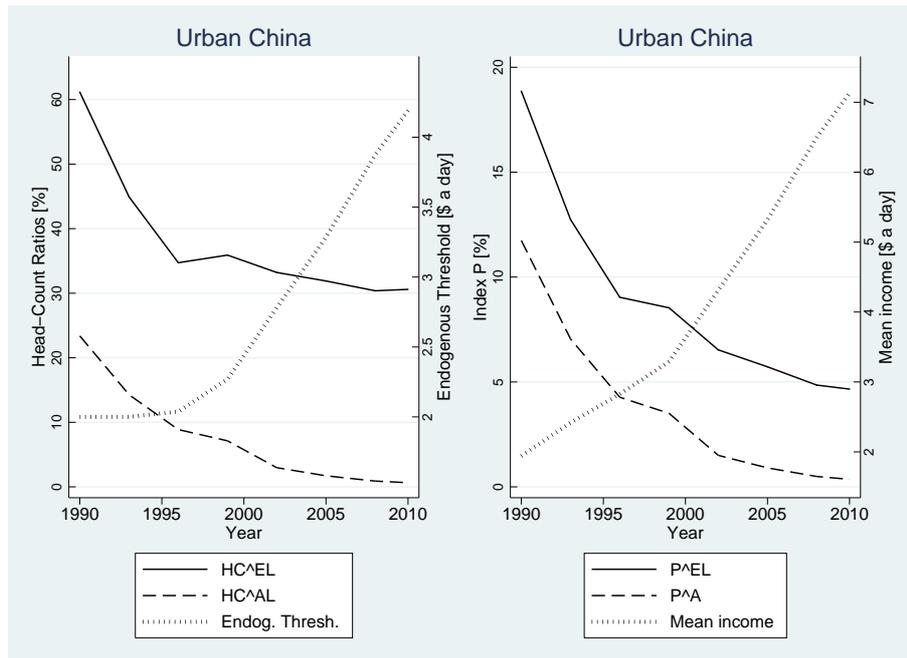
Figure 4: Evolution of income poverty between 1990 and 2010 in urban China as measured by P^{EL}. The left graph show the decomposition of poor individuals (HC^{EL}) between absolutely poor (HC^{AL}) and relatively poor. The right graph shows the decomposition of P^{EL} between the contribution of absolutely poor individuals (P^a) and that of relatively poor individuals (P^r). Source: PovcalNet.

6 Concluding remarks

Comparing income poverty between societies with different standards of living has always been done with extreme caution. This caution follows in part from the inability of standard indices – such as the popular Head-Count Ratio or Poverty Gap Ratio – to weigh in a plausible way the absolute and relative aspects of income poverty. I propose an alternative way to weigh these two aspects and use it to derive a new index. Because its individual poverty comparisons are more in line with standard intuitions, this new index provides a firmer foundation for poverty comparisons when standards of living differ.

There are several direct applications for this research. A prominent example is the measurement of income poverty by the World Bank. This institution recently established a commission aimed at advising it on the best way to monitor the realization of its twin goals.[25] The decomposition of the new index between absolute and relative poverty should simplify the analysis and the communication on the progress achieved towards its twin goals. In the same vein, the EU Commission could integrate a measure based on the new index into its AROPE measure. Also, countries whose official income poverty definition is judged non-

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[25]The Commission on Global Poverty was established in 2015. http://www.worldbank.org/en/programs/commission-on-global-poverty.

satisfactory could find interest in the new index. The United States constitute a prominent example as several observers like Ruggles (1990) and Citro and Michael (1995) questioned its absolute line. See Blank (2008) for a review of the political initiatives that have attempted to modify it.

Switching the poverty measure changes the evaluation of policies aimed at reducing poverty. Traditionally, policy makers use absolute measures for policy evaluation in low- and middle-income countries and relative measures in high-income countries. This practice ensures that the most relevant aspect of income poverty is captured in each case, at the cost of ignoring the other aspect. The limitation of this practice is that it yields extreme judgments on growth, as explained in the Introduction. The evaluation of policies with a measure based on the new index solves these limitations. As a consequence, the policies recommended by this index are in line with what the specific situation requires without being extreme.

More generally, the index proposed has applications outside income poverty measurement. If the emphasis has been put on income, the index can measure the poverty in any other resource for which both the absolute and relative aspects matter, like education or health.

7   Appendix

Proof of Theorem 1

Before proving each claim in turn, I study the restrictions that each property imposes on poverty indices in the extended FGT family.

Let the partial derivative of a function $f : \mathbb{R}^n \to \mathbb{R}$ in the direction i at point $x \in \mathbb{R}^n$ be denoted by $\partial_i f(x)$. *Monotonicity in Income* requires that for all $y \in Y$ and all $i \leq q$ we have:

$$\partial_i P(y) \leq 0.$$

Remember that $P(y) = \frac{1}{n} \sum_i d(y_i, \overline{y})$ and that we have $\partial_i \overline{y} = \frac{1}{n}$. By chain derivation, we have that *Monotonicity in Income* is satisfied if and only if for all $y \in Y$ and all $i \leq q$ we have:

$$\underbrace{\partial_1 d(y_i, \overline{y}) + \frac{1}{n} \sum_{j=1}^n \partial_2 d(y_j, \overline{y})}_{L_5} \leq 0. \tag{5}$$

Conditions (5) relies on partial derivatives of function d. Given its exponential form and the definition in two parts of e^r, we have that for all $y \in Y$ and all $i \leq q$ the partial derivative with respect to own income is:

$$\partial_1 d(y_i, \overline{y}) = -\frac{\alpha}{z(\overline{y}^r)} \left(1 - \frac{e^r(y_i, \overline{y})}{z(\overline{y}^r)}\right)^{1-\alpha} \qquad \text{if } y_i < z^a, \tag{6}$$

and[26]

$$\partial_1 d(y_i, \overline{y}) = -\frac{\alpha}{z(\overline{y}^r)} \left(1 - \frac{e^r(y_i, \overline{y})}{z(\overline{y}^r)}\right)^{1-\alpha} \frac{z(\overline{y}^r) - z^a}{z(\overline{y}) - z^a} \qquad \text{if } z^a \leq y_i. \tag{7}$$

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[26]If $y_i = z^a$, the partial derivative is not well-defined if $z(\overline{y}) \neq z(\overline{y}^r)$. I have defined in equation (7) this partial derivative to be $\partial_1 d(y_i, \overline{y}) := \lim_{x \to z^{a+}} \partial_1 d(x, \overline{y})$.

Regarding the partial derivative with respect to mean income, we have that for all $y \in Y$ and all $i \leq q$:

$$\partial_2 d(y_i, \overline{y}) = 0 \qquad \text{if } y_i \leq z^a, \tag{8}$$

and

$$\partial_2 d(y_i, \overline{y}) = -s(\overline{y})\frac{z(\overline{y}^r) - z^a}{z(\overline{y}) - z^a}\partial_1 d(y_i, \overline{y}) \qquad \text{if } z^a < y_i. \tag{9}$$

Observe that if $y_i = z(\overline{y})$ and $s(\overline{y}) > 0$, then $\partial_2 d(y_i, \overline{y}) > 0$.[27]

In turn, *Weak Transfer* requires that function d is convex in y_i at all $\overline{y} \geq 0$. That is, for all $y \in Y$ and any two $i, j \leq q$ with $y_i < y_j$ we have:[28]

$$\partial_1 d(y_i, \overline{y}) \leq \partial_1 d(y_j, \overline{y}). \tag{10}$$

Using conditions (5) and (10) and the expressions for the partial derivative of function d, I prove each claim in turn.

First, I prove claim 1: an index P in the extended FGT family satisfies *Monotonicity in Income* only if $\alpha = 1$. The proof is case by case.

- Case 1: $0 < \alpha < 1$.

  Consider any $y \in Y$ with $q \geq 2$ and $\overline{y} = \overline{y}^*$ such that $z(\overline{y}^*) \leq \overline{y}^*$ and $s(\overline{y}^*) > 0$. As the line z is monotonic, this $\overline{y}^*$ exists. When the income of individual 1 increases marginally, we have that

  $$L_5(y) \geq \partial_1 d(y_1, \overline{y}) + \partial_2 d(y_q, \overline{y}),$$

  because $\partial_2 d(y_j, \overline{y}) \geq 0$ for all $j < q$. As $\alpha < 1$ and $s(\overline{y}^*) > 0$, we have from equations (9) and (7) that

  $$\lim_{x \to z(\overline{y}^*)^-} \partial_2 d(x, \overline{y}) = \infty.$$

  For a fixed y_1, there exists hence a value of y_q sufficiently close to $z(\overline{y}^*)$ such that

  $$L_5(y) \geq \partial_1 d(y_1, \overline{y}) + \partial_2 d(y_q, \overline{y}) > 0,$$

  and hence condition (5) is violated. Hence, *Monotonicity in Income* does not hold.

- Case 2: $1 < \alpha$.

  This case is plagued with the reversed problem. Consider any $y \in Y$ with $\overline{y} = \overline{y}^*$, $q \geq 2$ and $y_{q-1} > z^a$. Last requirement implies that individual $q-1$ is relatively poor. When the income of individual q increases marginally, we have again

  $$L_5(y) \geq \partial_1 d(y_q, \overline{y}) + \partial_2 d(y_{q-1}, \overline{y}).$$

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[27] Its expression is defined in equation (9) as $\lim_{x \to z(\overline{y})^-} \partial_2 d(x, \overline{y})$.

[28] If $y_j = z^a$, the definition of the partial derivative adopted above by equation (7) is not relevant. In this particular case, the value returned by equation (6), corresponding to $\partial_1 d(y_j, \overline{y}) := \lim_{x \to z^{a-}} \partial_1 d(x, \overline{y})$, is relevant.

As $\alpha > 1$, we have from equation (7) that

$$\lim_{x \to z(\overline{y}^*)^-} \partial_2 d(x, \overline{y}) = 0.$$

As $y_{q-1} > z^a$ and $s(\overline{y}^*) > 0$, we have that $\partial_2 d(y_{q-1}, \overline{y}) > 0$. Again, for y_q sufficiently close to $z(\overline{y}^*)$, we have

$$L_5(y) \geq \partial_1 d(y_1, \overline{y}) + \partial_2 d(y_q, \overline{y}) > 0,$$

and condition (5) is violated. Hence, *Monotonicity in Income* does not hold.

- Case 3: $\alpha = 0$.

  Index P is the Head-Count Ratio. *Monotonicity in Income* is violated for any $y \in Y$ with $q < n$ and $\overline{y} = \overline{y}^*$ and one non-poor individual i has income $y_i = z(\overline{y})$, as shown in Table 1.

Second, I prove claim 2. I show first that an index P in the extended FGT family with $\alpha = 1$ satisfies *Weak Transfer* if and only if $\overline{y}^r = 0$. When $\alpha = 1$, equations (6) and (7) simplify. We have that for all $y \in Y$ and all $i \leq q$:

$$\partial_1 d(y_i, \overline{y}) = -\frac{1}{z(\overline{y}^r)} \qquad \text{if } y_i < z^a, \tag{11}$$

and[29]

$$\partial_1 d(y_i, \overline{y}) = -\frac{1}{z(\overline{y}^r)} \frac{z(\overline{y}^r) - z^a}{z(\overline{y}) - z^a} \qquad \text{if } z^a \leq y_i. \tag{12}$$

From equations (11) and (12), condition (10) is satisfied, and hence *Weak Transfer*, if and only if we have for all $\overline{y} \geq 0$ that

$$z(\overline{y}^r) \leq z(\overline{y}).$$

Given line z is monotonic, this implies $\overline{y}^r = 0$. Observe that if z is such that $s(\overline{y}) = 0$ for all $\overline{y} \in [0, a)$ with $a > 0$, then taking any $\overline{y}^r \in [0, a)$ yields a poverty index equivalent to that obtained when selecting $\overline{y}^r = 0$.

Finally, I show that if P is a poverty index in the extended FGT family with $\alpha = 1$ and $\overline{y}^r = 0$, then it satisfies both properties. I have just shown that P satisfies *Weak Transfer* since $\overline{y}^r = 0$. There remains to show that P satisfies *Monotonicity in Income*.

Condition (5) can be simplified using equation (8). *Monotonicity in Income* is satisfied if and only if for all $y \in Y$ and all $i \leq q$ we have:

$$\partial_1 d(y_i, \overline{y}) + \frac{1}{n} \sum_{j=q^a+1}^{n} \partial_2 d(y_j, \overline{y}) \leq 0. \tag{13}$$

As $\overline{y}^r = 0$ and z is monotonic, equations (11) and (12) imply that for all $i \leq q$ and all $j \leq q$ that are relatively poor ($q^a < j \leq q$) we have that

$$\partial_1 d(y_i, \overline{y}) \leq \partial_1 d(y_j, \overline{y}).$$

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[29] The same remark as the one made before equation (7) applies.

As by assumption $s(\overline{y}) \leq 1$ for all $\overline{y} \geq 0$, equation (9) implies that for all j with $q^a < j \leq q$ we have that

$$\partial_1 d(y_j, \overline{y}) \leq -\partial_2 d(y_j, \overline{y}).$$

The two last inequalities implies together that for all $i \leq q$ and $q^a < j \leq q$ we have that

$$\partial_1 d(y_i, \overline{y}) \leq -\partial_2 d(y_j, \overline{y}).$$

Since $n \geq n - q^a$, we have that condition (13) holds. Hence P satisfies both properties.

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