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Targeted campaign competition, loyal voters, and supermajorities*

Pierre C. Boyer† and Kai A. Konrad‡

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Abstract

We consider a two-candidate campaign competition in majoritar-
ian systems with many voters. Some voters are loyal, some can be
influenced by campaign spending. Own loyalty with respect to a can-
didate is the voter's private information. Candidates simultaneously
choose their campaign budgets and how to allocate them among the
voters. We show that a candidate who has a group of loyal voters
wins with a higher probability, but chooses the same expected budget
size as the rival candidate. The equilibrium distributions of campaign
spending target all voters equally in expectation, but target some vot-
ers more than others ex post.

Keywords: Campaign competition; vote buying; supermajorities;
targeting; flexible budgets; asymmetric information.

JEL classification code: D72; D78; D82.

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1 Introduction

Campaign spending is an important aspect of electoral competition.[1] It has often been formalized as a vote-buying activity, much like in an all-pay contest. Seminal contributions include Erikson and Palfrey (2000), Che and Gale (1998), Sahuguet and Persico (2006), and Meirowitz (2008).[2] Much of this literature considers the electorate as a homogeneous group or assumes that all voters receive the same campaign treatment by the candidates. This allows previous papers to focus on the choice of the total amount of spending, which then translate into candidates' vote shares. However, this restriction is becoming a concern as microtargeting of campaign spending at the level of individual voters is becoming the norm.[3] Targeting, or more generally, the allocation of campaign spending on different voters or voter groups is

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[1]The amounts expended on electoral campaigns are sizeable, and seemingly increasing. According to the New York Times, for instance, campaign spending for Barack Obama and Mitt Romney were $ 985.7m and $ 992.0m in 2012 (see http://elections.nytimes.com/2012/campaign-finance, as viewed on May 27, 2014). As Meirowitz (2008) points out, precise policy statements play a minor role in this type of persuasive campaign spending.

[2]The all-pay contest nature of electoral competition has also been stressed in many other important contributions. These include Skaperdas and Grofman (1995), Diermeier and Myerson (1999), and Pastine and Pastine (2012). See also Snyder (1989) for an earlier study on political campaign expenditures.

[3]A feature story of the MIT Technology Review in 2012 describes the increasing availability and usage of microtargeting in the last two US presidential elections (see http://www.technologyreview.com/featuredstory/509026/how-obamas-team-used-big-data-to-rally-voters, as viewed on July 21, 2014). Kenski, Hardy and Jamieson (2010) and Ridout, Franz, Goldstein, and Feltus (2012) highlight the role of microtargeting in the media in the US presidential elections. Jamieson (2013) envisions a change in electoral competition from a focus on swing states to swing individuals. Hersh and Schaffner (2013) highlight that partisans and candidates increasingly target their campaign contributions at the individuals' level.

less carefully studied in formal analyses.[4] Politicians and their campaign managers have to decide not only how much campaign money to mobilize and spend, but they also have to decide how to spend the money. This dual choice is at the core of our analysis where politicians decide not only about their budgets for campaign spending but also about the allocation of their budgets among voters.

Allowing targeting of campaign spending to take place at the level of individual voters raises questions about the information available to candidates about characteristics or types of voters. Voters typically differ in their loyalty to candidates. Since politicians' objective during electoral campaigns is to insure the support of the voters for the upcoming election, channeling scarce resources to the voters that can be convinced is the politicians' Holy Grail.[5] We capture this important aspect by introducing some information asymmetry between voters and candidates about individual voters' loyalty.[6]

More specifically, we study a majoritarian electoral competition between two candidates who choose simultaneously their budgets for campaign spending and how to allocate their budgets among a large number of voters. The voters may differ in their loyalty to candidates and candidates may be incompletely informed about individual voters' loyalty.

The characterization of the Nash equilibrium of our electoral game leads to the following main results: (1) Loyalty advantages lead to an equilib-

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[4]An important exception is Sahuguet and Persico (2006). We clarify the differences with our paper below.

[5]The Obama 2012 campaign used big data techniques and experimental methods to try to identify the voters most likely to change their views about the candidates after hearing Obama's arguments and then channel resources to such type of voters (see http://www.technologyreview.com/featuredstory/509026/how-obamas-team-used-big-data-to-rally-voters, as viewed on July 21, 2014).

[6]Heterogeneity of voters along this dimension and issues of incomplete information have been assessed empirically, see Basinger and Lavine (2005), for instance.

rium in which the advantaged candidate wins with a higher probability, but chooses the same expected budget size as the rival candidate. (2) While a candidate treats voters symmetrically from an ex-ante point of view, the ex-post transfers offered to different voters have considerable variance. Some voters are the targets of much campaign spending, others receive little. (3) A similar variability emerges in the equilibrium for campaign budget choices, due to the non-existence of an equilibrium in pure strategies. (4) When the loyalty of voters is observed by candidates only buyable voters get positive transfers in expectation, however, the expected budgets for campaign and the allocation of budgets among the buyable voters are identical to the case with private information. (5) The second and third result jointly lead to major outcome heterogeneity in vote shares. Hence, the equilibrium which we consider endogenously generates large supermajorities, even if candidates and voters are perfectly symmetric ex ante.

In a majoritarian system, a politician's payoff has a sharp discontinuity at 50 percent of the votes. Rather than allocating a given campaign budget homogeneously among the whole set of voters, this discontinuity makes it worthwhile to focus on the votes of a majority group.[7] The first systematic and seminal formal analysis of the targeting of individual voters in electoral competition in a majoritarian system is by Groseclose and Snyder (1996, 2000) and Banks (2000). They allow for a large, typically finite set of voters, such as in a committee or a legislature. Voters differ in their inclination to vote for one or the other candidate. In their framework the preferences of individual voters are known to the political candidates. A key assumption in their analysis is that candidates choose their vote-buying efforts sequentially, with the incumbent choosing first, followed by a challenger.[8] They show

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[7]Jaryczower and Mattozzi (2013) discuss the importance of this discontinuity for electoral competition in majoritarian systems and compare it with plurality systems.

[8]Dekel, Jackson, and Wolinsky (2008, 2009) extend this analysis by allowing the can-

that this set-up typically generates supermajorities: rather than buying just a tiny majority of votes, in the Stackelberg equilibrium the incumbent tends to 'buy' a supermajority.[9]

A sequential order of moves with the challenger moving second is to the advantage of the challenger, but to the disadvantage of the incumbent. There might be instances when a challenger may nevertheless be in this advantageous position. However, as both players would like to choose their targets last and prefer to keep own budget allocation choices secret and not give the competitor an easy way to react to their choices, a simultaneity of choices may be more frequently the case. In many contexts where our model may be applied, such as for juries, vote buying may be an illegal, or an illegitimate action, such that the bids are preferably made in private/secretly. Simultaneity of the two candidates' choices is also a plausible outcome in such cases. Simultaneity of choices changes the structure of equilibrium dramatically. Pure strategy equilibrium vanishes, and in the Nash equilibrium candidates need to rely on mixed strategies when choosing their budgets and their budget allocation rules. We focus on a situation in which both competitors must make their budget choices not knowing the budget size and the allocation rule chosen by the competitor. In the resulting Nash equilibrium they will

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didates to alternate in making vote-buying offers until they no longer want to make new offers to voters.

[9]Their result has a nice intuition. The strategically disadvantaged candidate who makes his vote purchases first anticipates the vote-purchasing options of the follower candidate. To make the life of the follower candidate harder it may be worthwhile to buy a supermajority of voters. If the leader just bribed 50.1 percent of the voters, the follower could purchase the votes of 49.9 percent of the voters for almost nothing and would only have to expend money to purchase a very small margin of less than 0.2 percent of the voters who received an offer by the leader candidate. If instead, the leader candidate made payments to 70 percent of voters, then the follower would get 30 percent of the votes for a very low price, but the follower would need to buy almost 20.1 percent of votes for a high price. The logic of this argument strongly relies on sequential choice.

have expectations about the actions of their competitor, but none of them will have the strategic advantage of the follower who can see which votes are actually cheapest to buy. Simultaneity not only triggers higher formal complexity, but it also addresses an empirically highly relevant case. The Nash equilibrium is characterized by supermajorities which emerge with a very high probability. Hence, the analysis offers an explanation for the emergence of supermajorities. Here, supermajorities are caused by a different mechanism and follow a different pattern.

Our contribution is also indirectly related to another large literature that focuses on electoral promises. Drawing on the tradition of Colonel Blotto games,[10] Myerson (1993) considers a continuum of voters and two candidates who announce platforms that specify whom to tax and whom to give money to. Lizzeri and Persico (2001) broaden this perspective by allowing candidates to use the tax revenue to bribe voters or to spend it on public goods. Kovenock and Roberson (2009) and Crutzen and Sahuguet (2009) allow for inefficiencies in this process of reallocating or collecting resources, respectively. We adopt some features from this literature (a large group of voters, simultaneous choices made by the candidates, differentiating and targeting net transfers to each single voter), but we depart from it, allowing the candidates to choose the size of their budgets. The paper also offers new theoretical insights that contribute to this literature. It departs from the common-knowledge assumption about voters' genuine preferences. Rather, voters are heterogeneous in their loyalties. Candidates know about the distribution of these loyalty types, but do not know the type of each individual voter. Incomplete information of this kind affects targeting in the

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[10]Early contributions to this literature are Gross and Wagner (1950) and Shubik (1970). A most encompassing treatment of the problem is by Roberson (2006). Kovenock and Roberson (2008) apply this theory in the context of electoral competition. Barelli, Govindan and Wilson (2014) address the same structural problem on a more general level. For a survey see Kovenock and Roberson (2012).

equilibrium. Incomplete information about voters' type is also assumed in Bierbrauer and Boyer (2014). They study the political outcome in classical frameworks in normative public economics (publicly provided goods and redistributive taxation). Their and our framework have in common that candidates are ignorant about the individual voters' types, let it be either preferences over policies as in Bierbrauer and Boyer (2014), or loyalty towards candidates as in our case.

Sahuguet and Persico (2006) consider a two-stage game, where the second stage is similar to a model of redistributional politics in the spirit of Myerson (1993), but with proportional representation. Prior to this stage, candidates can invest effort that increases their perceived valence. This valence increases the budget available for redistribution in the second stage since the more valent candidate needs fewer resources to bribe voters. Their first-period choices are mutually observed and are common knowledge at the second stage. Proportional representation avoids the important discontinuity in payoffs that is generated by a majoritarian system. Our paper instead looks at a majoritarian election with a simultaneous choice of budget and budget allocation rule.

We proceed as follows. Section 2 describes the formal framework. Section 3 solves for the Nash equilibrium. Section 4 presents a discussion of the role of private information about loyalty in political campaigns and shows how our framework generates supermajorities. The last section contains concluding remarks. We also relegate some of our proofs to the Appendix.

2 The formal framework

Let there be an electorate/legislature/jury/committee which consists of a continuum of voters of measure 1.[11] The members i of the set choose between two candidates by majority voting. One candidate is denoted by A, the other by B. Each candidate would like to win the election. Each candidate may try to influence the decision outcome. Candidates A and B choose a vote-buying budget, measured in money and denoted by $a \geq 0$ and $b \geq 0$, respectively. The quantities a and b are chosen independently. The candidates may use mixing and can choose random distributions from which the actual a and b are drawn. We denote the cumulative distribution functions of these random distributions as $\Phi(a)$ and $\Gamma(b)$. The vote-buying candidates must also decide how to allocate these budgets among the different voters. Each voter can receive different amounts of campaign spending. We denote by x_i and y_i the amounts received by voter i from candidate A and B, respectively. For any budget a and b, the vote-buying candidates choose the amounts of the bribes awarded to each given voter. Candidates can use mixing on these bribes and can choose random distributions from which x_i and y_i are drawn. We denote the cumulative distribution functions of these random distributions as F_a and G_b where the subscript refers to the budget a and b, respectively. The budget constraints require that the mean of the distributions F_a and G_b is equal to the budgets such that

$$\int_0^{+\infty} x dF_a(x) = a \text{ and } \int_0^{+\infty} y dG_b(y) = b.$$

A strategy of candidate A consists of a choice of (Φ, F_a): a mixed strategy on the budget size, and a set of distribution functions F_a that describes

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[11]The application of our model to jury or committee may be sensible to the assumption that the population of voters is large. However, the technical difficulties raised by a small electorate makes to it useful to get, as a first approximation, qualitative results in the case where the jury or committee has many members.

the distribution of bribes for each possible budget a. Similarly, a strategy of candidate B consists of an analogous pair (Γ, G_b). All choices by both candidates take place simultaneously, where the distribution functions Φ, Γ F_a and G_a are from the set of feasible cumulative distribution functions. Representatives of actions chosen from these sets will be denoted (Φ, F_a) and (Γ, G_b) for A and B, respectively. This completes the description of the candidates' strategy space.

Note that this description logically decomposes the vote-buying decision into two: one decision is on the expected amount of money to be used. We refer to it as the *budget choice*. A second decision is on how to allocate this money among the voters. We refer to it as the *budget allocation choice*. This provides a useful logic structure to the decision problem and it also has its mirror image in the characterization of equilibrium.

Neither the budgetary choices nor the budget allocation choices of one candidate can be observed by the other candidate prior to this candidate's own choices. From a strategic point of view, both the budget choices and the budget allocation choices take place simultaneously such that the relevant equilibrium concept is Nash equilibrium, and not subgame perfect equilibrium.

Vote buying may affect the choices of some voters more than the choices of other voters. We assume that each single voter is either a loyal voter for candidate A, or a voter who can be bought.[12] A loyal voter always votes faithfully for candidate A, irrespective of any vote-buying activities. Loyalty for party A is independently and identically distributed across voters and each voter i is a loyal voter for party A with probability $\Delta \in [0, \frac{1}{2})$.[13] With

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[12]It saves notation if we consider the case in which only candidate A has loyal voters. But the analysis can be extended with some notational effort if both candidates have shares of loyal voters.

[13]If $\Delta \geq \frac{1}{2}$ the problem degenerates and becomes uninteresting. The pivotal voter would be loyal in this case, and any vote-buying effort would be fully wasted. Vote buying and

probability $1 - \Delta$ the voter is buyable. If the voter is buyable the voter votes for A if $x_i > y_i$, for B if $y_i > x_i$, and randomizes symmetrically if $x_i = y_i$. Hence, by the law of large numbers, the candidates know the aggregate share Δ of voters who cannot be bought. However, whether a specific voter i belongs to the buyable group or to the group of loyal voters cannot be observed by the vote-buying candidates when they make their budget allocation choices. A voter who cannot be bought takes the money but votes for the candidate he or she prefers, irrespective of the x_i and y_i received. Whether a voter is loyal or can be bought is known to the voter.

The assumed choice behavior of voters respects and maps an important time-consistency problem: voters may take the bribes and then still follow their genuine preferences. In this case, suppose that a set Δ of voters strictly prefer candidate A, whereas the set of voters with measure $1 - \Delta$ is just indifferent between the two candidates. Time-consistent voting behavior suggests that loyal voters do not reveal their loyalty. They just take the payments (from both sides) and then cast their own vote according to their genuine preference. Vote-buying payments to voters in the subset Δ would have no influence on their voting choices. As will become clear when loyalty is observable, candidates who would like to save on their resources prefer not to make payments to these voters. But it may be impossible for them to identify and distinguish this group from buyable voters, i.e., from voters whose voting choice is a non-trivial function of vote-buying. This latter group are the voters who are indifferent between A and B. For them, the comparison of payments x_i and y_i triggers their candidate choice.

Finally, we turn to the preferences of the vote-buying candidates. Each candidate attributes a prize value of $v = 1$ to winning the election. This value may be thought of as the benefit of being enabled to extract or divert resources for their own benefit. Accordingly, the actual payoff of A who

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then budget choices would be inconsequential for the majority outcome.

expends amount a on vote buying is

$$
v_A = \begin{cases} 1 - a & \text{if} \quad \sigma_A > \frac{1}{2} \\ \frac{1}{2} - a & \text{if} \quad \sigma_A = \frac{1}{2} \\ -a & \text{if} \quad \sigma_A < \frac{1}{2} \end{cases}
$$

where σ_A is the share of voters voting for A. The payoff v_B is defined analogously.

3 Equilibrium characterization

This section characterizes the equilibrium budget choices and budget allocation rules. Intuitively, we expect that the existence of loyal voters generates an advantage for A, even under the conditions of incomplete information. The following theorem shows how this advantage translates into equilibrium vote-buying budgets, distributions of payments, and winning probabilities.

Theorem 1 *The choices of* $(\Phi, F_a), (\Gamma, G_b)$ *with*

$$
F_a(x) = \begin{cases} 1 - \frac{1-2\Delta}{1-\Delta} + \frac{1-2\Delta}{1-\Delta} \frac{x}{2a\frac{1-\Delta}{1-2\Delta}} & \text{for} \quad x \in [0, 2a\frac{1-\Delta}{1-2\Delta}] \\ 1 & \text{for} \quad x > 2a\frac{1-\Delta}{1-2\Delta} \end{cases} \quad , \tag{1}
$$

$$
G_b(y) = \begin{cases} \frac{y}{2b} & \text{for} \quad y \in [0, 2b] \\ 1 & \text{for} \quad y > 2b \end{cases} \quad , \tag{2}
$$

$$
\Phi(a) = \begin{cases} \frac{a}{\frac{1-2\Delta}{1-\Delta}} & \text{for} \quad a \in [0, \frac{1-2\Delta}{1-\Delta}) \\ 1 & \text{for} \quad a \geq \frac{1-2\Delta}{1-\Delta} \end{cases} \quad , \tag{3}
$$

and

$$
\Gamma(b) = \begin{cases} 1 - \frac{1-2\Delta}{1-\Delta} + \frac{b}{\frac{1-\Delta}{1-2\Delta}} & \text{for} \quad b \in [0, 1) \\ 1 & \text{for} \quad b \geq 1 \end{cases} \tag{4}
$$

constitute a Nash equilibrium. Expected campaign budgets are

$$
Ea = Eb = \frac{1}{2}\frac{1-2\Delta}{1-\Delta}. \tag{5}
$$

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Expected payoffs in this equilibrium are

$$Ev_A = \frac{\Delta}{1-\Delta} \text{ and } Ev_B = 0, \tag{6}$$

and winning probabilities are

$$q_A = \frac{1}{2}\frac{1}{1-\Delta} \text{ and } q_B = \frac{1}{2}\frac{1-2\Delta}{1-\Delta}. \tag{7}$$

Proof The structure of the proof follows steps (I) to (V): (I) We consider the optimality of G_b as in (2) for a given b, given that the budget a is drawn from a distribution $\Phi(a)$ and is allocated according to the rule (1). (II) We consider the optimality of (1) for a given a if candidate B chooses (G_b, Γ). (III) We draw conclusions about the candidates' payoffs as functions of (a, b). (IV) We use these results to confirm the optimality of the choices Φ and Γ. (V) We show that A and B expend the same budget in expectation, that A has a higher payoff and a higher probability of winning than B if $\Delta > 0$ in the equilibrium.

Step (I): We show that G_b maximizes B's winning probability for a given b, for all values of *(i)* $a = \frac{1-2\Delta}{1-\Delta}b$, *(ii)* $a > \frac{1-2\Delta}{1-\Delta}b$, and *(iii)* $a < \frac{1-2\Delta}{1-\Delta}b$.

(i) We start with the case $a = \frac{1-2\Delta}{1-\Delta}b$. As $F_a(x) = 1$ for $x \geq 2a\frac{1-\Delta}{1-2\Delta}$, the allocation rule F_a induces a vote share for B that is equal to

$$\sigma_B = (1-\Delta)\left[\int_0^{2a\frac{1-\Delta}{1-2\Delta}} F_a(x)\, dG_b(x) + \int_{2a\frac{1-\Delta}{1-2\Delta}}^{+\infty} dG_b(x)\right],$$

since the share of the vote of candidate B is equal to the probability that any random voter, among the non-loyal voters, receives an offer from B which is higher than the offer he receives from A.

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Inserting (1) and using $dG_b(y) = 0$ for all $y > 2b = 2\, a\frac{1-\Delta}{1-2\Delta}$, we find

$$
\begin{aligned}
\sigma_B &= (1-\Delta)\left[\int_0^{2b}(1 - \frac{1-2\Delta}{1-\Delta} + \frac{1-2\Delta}{1-\Delta}\frac{x}{2a\frac{1-\Delta}{1-2\Delta}})\, dG_b(x)\right] \\
&= (1-\Delta)\left[1 - \frac{1-2\Delta}{1-\Delta} + \frac{1-2\Delta}{1-\Delta}\frac{1}{2a\frac{1-\Delta}{1-2\Delta}}\int_0^{2b} x\, dG_b(x)\right] \\
&= (1-\Delta)\left[1 - \frac{1-2\Delta}{1-\Delta} + \frac{1-2\Delta}{1-\Delta}\frac{b}{2a\frac{1-\Delta}{1-2\Delta}}\right] = \frac{1}{2}\ ,
\end{aligned}
$$

where the equality in the third line follows from the budget constraint, and the last equality follows from $b = a\frac{1-\Delta}{1-2\Delta}$.

We now show that $\sigma_B = \frac{1}{2}$ is the maximum vote share $\hat{\sigma}_B$ that can be reached for any feasible $\hat{G}_b$ with an expected budget $b = a\frac{1-\Delta}{1-2\Delta}$. Note that

$$
\begin{aligned}
\hat{\sigma}_B &= (1-\Delta)\left[\int_0^{+\infty} F_a(x)\, d\hat{G}_b(x)\right] \\
&\leq (1-\Delta)\left[1 - \frac{1-2\Delta}{1-\Delta} + \frac{1-2\Delta}{1-\Delta}\frac{1}{2a\frac{1-\Delta}{1-2\Delta}}\int_0^{+\infty} x\, d\hat{G}_b(x)\right] = \frac{1}{2}\ ,
\end{aligned}
$$

where the inequality in the second line is strict whenever $x > 2a\frac{1-\Delta}{1-2\Delta}$ has positive mass. The last equality follows from the budget constraint and from $b = a\frac{1-\Delta}{1-2\Delta}$.

Accordingly, G_b yields the maximum vote share that is equal to $\frac{1}{2}$ and leads to a winning probability $q_B = \frac{1}{2}$.

(ii) Now we turn to the case $a > \frac{1-2\Delta}{1-\Delta}b$. We show that G_b is also optimal for this range of budgets for candidate A. Note that

$$
\sigma_B = (1-\Delta)\left[\int_0^{2a\frac{1-\Delta}{1-2\Delta}} F_a(x)\, dG_b(x)\right] < \frac{1}{2},
$$

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and, hence, $q_B = 0$. To show the optimality of G_b in this range it is sufficient to show that no other $\hat{G}_b(x)$ exists that yields $\hat{\sigma}_B \geq \frac{1}{2}$. Consider

$$
\begin{aligned}
\hat{\sigma}_B &= (1 - \Delta) \left[\int_0^{+\infty} F_a(x) \, d\hat{G}_b(x) \right] \\
&\leq (1 - \Delta) \left[1 - \frac{1 - 2\Delta}{1 - \Delta} + \frac{1 - 2\Delta}{1 - \Delta} \frac{1}{2a\frac{1-\Delta}{1-2\Delta}} \int_0^{+\infty} x \, d\hat{G}_b(x) \right] < \frac{1}{2} \,,
\end{aligned}
$$

where the inequality in the second line is strict whenever $x > 2a\frac{1-\Delta}{1-2\Delta}$ has positive mass, the last strict inequality follows from the budget constraint and holds for all $a\frac{1-\Delta}{1-2\Delta} > b$. We conclude that $q_B = 0$ for all $a > \frac{1-2\Delta}{1-\Delta}b$ and all $\hat{G}_b(x)$ with an expected budget b.

(iii) Now we turn to the case $a < \frac{1-2\Delta}{1-\Delta}b$. In this case, inserting (1) and (2) yields

$$
\begin{aligned}
\sigma_B &= (1 - \Delta) \left[\int_0^{2a\frac{1-\Delta}{1-2\Delta}} F_a(x) \, dG_b(x) + \int_{2a\frac{1-\Delta}{1-2\Delta}}^{+\infty} dG_b(x) \right] \\
&= (1 - \Delta) \left[\int_0^{2a\frac{1-\Delta}{1-2\Delta}} (1 - \frac{1-2\Delta}{1-\Delta} + \frac{1-2\Delta}{1-\Delta} \frac{x}{2a\frac{1-\Delta}{1-2\Delta}}) \frac{1}{2b} dx + \int_{2a\frac{1-\Delta}{1-2\Delta}}^{2b} \frac{1}{2b} dx \right] \\
&= (1 - \Delta) \left[\frac{1-2\Delta}{1-\Delta} \int_0^{2a\frac{1-\Delta}{1-2\Delta}} (\frac{x}{2a\frac{1-\Delta}{1-2\Delta}} - 1) \frac{1}{2b} dx + \frac{1}{2b} \int_0^{2b} dx \right] \\
&= (1 - \Delta) \left[-\frac{a}{2b} + 1 \right] \\
&> (1 - \Delta) \left[1 - \frac{\frac{1-2\Delta}{1-\Delta}b}{2b} \right] = \frac{1}{2}.
\end{aligned}
$$

The strict inequality in the last line holds because $a < \frac{1-2\Delta}{1-\Delta}b$. Now, $\sigma_B > \frac{1}{2}$ implies $q_B = 1$, and this is the maximum winning probability. It cannot be increased by a different $\hat{G}_b(x)$. This concludes the first part of the proof.

Step (II): We now turn to the optimality of the allocation rule (1) for a given a and given that b is drawn from a distribution $\Gamma(b)$ and is allocated according to the rule (2). We confirm this again in three parts, showing that F_a is an optimal reply if $b = \frac{1-\Delta}{1-2\Delta}a$, if $b > \frac{1-\Delta}{1-2\Delta}a$ and if $b < \frac{1-\Delta}{1-2\Delta}a$.

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The share of the vote of candidate A is equal to the share of loyal voters plus the probability that any random voter, among the non-loyal voters, receives an offer from A which is higher than the offer he receives from B. Thus,

$$\sigma_A = \Delta + (1 - \Delta)\left[\int_0^{+\infty} G_b(x)dF_a(x)\right]$$

It is easy to find that, applying the equilibrium actions, (1) and (2) imply that

$$\sigma_A > \frac{1}{2} \text{ if } b < \frac{1-\Delta}{1-2\Delta}a \text{ , implying } q_A = 1,$$

$$\sigma_A = \frac{1}{2} \text{ if } b = \frac{1-\Delta}{1-2\Delta}a, \text{ implying } q_A = \frac{1}{2}, \text{ and}$$

$$\sigma_A < \frac{1}{2} \text{ if } b > \frac{1-\Delta}{1-2\Delta}a, \text{ implying } q_A = 0.$$

We now consider for $b = \frac{1-\Delta}{1-2\Delta}a$, for $b > \frac{1-\Delta}{1-2\Delta}a$ and for $b < \frac{1-\Delta}{1-2\Delta}a$ if there is an $\hat{F}_a(x)$ that yields a higher winning probability.

(i) For $b < \frac{1-\Delta}{1-2\Delta}a$, any $\hat{F}_a(x)$ could not increase the winning probability further, as it is already equal to 1.

(ii) For $b = \frac{1-\Delta}{1-2\Delta}a$ the allocation rule (1) induces a vote share for A that is equal to $\frac{1}{2}$ and a winning probability $q_A = \frac{1}{2}$. We ask if there exists $\hat{F}_a(x)$ that yields a vote share that exceeds $\frac{1}{2}$. The vote share is

$$\hat{\sigma}_A = \Delta + (1 - \Delta)\left[\int_0^{+\infty} G_b(x)d\hat{F}_a(x)\right]$$

$$\leq \Delta + (1 - \Delta)\left[\int_0^{+\infty} \frac{x}{2b}d\hat{F}_a(x)\right]$$

$$= \Delta + (1 - \Delta)\frac{1}{2}\frac{1-2\Delta}{1-\Delta} = \frac{1}{2},$$

for all feasible $\hat{F}_a$.

(iii) For $b > \frac{1-\Delta}{1-2\Delta}a$ the allocation rule (1) induces a vote share for A that is smaller than $\frac{1}{2}$ and a winning probability $q_A = 0$. We ask again if there

exists $\hat{F}_a(x)$ that yields a positive winning probability. The vote share is

$$
\begin{aligned}
\hat{\sigma}_A \;&=\; \Delta + (1-\Delta)\left[\int_0^{+\infty} G_b(x)d\hat{F}_a(x)\right] \\
&\leq\; \Delta + (1-\Delta)\left[\int_0^{+\infty} \frac{x}{2b}\,d\hat{F}_a(x)\right] \\
&=\; \Delta + (1-\Delta)\frac{a}{2b} \\
&<\; \Delta + (1-\Delta)\frac{a}{2\frac{1-\Delta}{1-2\Delta}a} = \frac{1}{2}.
\end{aligned}
$$

The last strict inequality comes from $b > \frac{1-\Delta}{1-2\Delta}a$. This completes Step (II).

Step (III): The proof in (I) and (II) identified a hyperplane

$$
a(b) = \frac{1-2\Delta}{1-\Delta}b,
$$

which separates the $a - b$ space such that

$$
q_A(a,b) = \begin{cases} 1 & \text{if} \quad a > a(b) \\ \frac{1}{2} & \text{if} \quad a = a(b) \\ 0 & \text{if} \quad a < a(b) \end{cases},
$$

and $q_B = 1 - q_A$.

Step (IV): We can solve the problem of choosing a and b as a reduced game with payoffs $Ev_A = q_A - a$ and $Ev_B = q_B - b$ that is equivalent to an all-pay auction in all-pay bids a and b. This problem is equivalent to Konrad (2002) and, in the context of campaign competition, to Meirowitz (2008). The equilibrium solution is (3) and (4).

We confirm that (3) and (4) are mutually optimal replies as follows. Consider first the optimality of $\Phi(a)$ given $\Gamma(b)$. Candidate B will not expend an expected budget that exceeds $b = 1$ and does not have a mass point on $b = 0$. Hence, candidate A wins with probability 1 for all $a > a(b)$. Given

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$\Gamma(b)$, A's payoff as a function of a in the range $a \in (0, \frac{1-2\Delta}{1-\Delta})$ is

$$Ev_A(a) = \Gamma(b(a)) - a = 1 - \frac{1-2\Delta}{1-\Delta} = \frac{\Delta}{1-\Delta} \qquad (8)$$

where $b(a) = \frac{1-\Delta}{1-2\Delta}a$ is the inverse function of $a(b)$. This shows that any $a \in (0, \frac{1-2\Delta}{1-\Delta})$ yields the same expected payoff for A. Hence, a random choice according to $\Phi(a)$ is optimal for A.

Similarly, the payoff of B as a function of b for a given $\Phi(a)$ is

$$Ev_B(b) = \Phi(a(b)) - b = \frac{1-\Delta}{1-2\Delta}a(b) - b = \frac{1-\Delta}{1-2\Delta}\frac{1-2\Delta}{1-\Delta}b - b = 0. \qquad (9)$$

Hence, a random choice according to $\Gamma(b)$ is optimal for B.

Step (V): As a final step we calculate the expected budgets, expected winning probabilities, and expected payoffs in the equilibrium. Equation (5) follows directly from (3) and (4). Expected payoffs are calculated in (8) and (9). Randomization according to (3) and (4) implies that the probability for $a = a(b)$ is zero. Hence, the probability that A will win is equal to the probability that $a > a(b)$. It follows from (3) and (4) that this probability is equal to equation (7). ∎

The proof of Theorem 1 combines two elements. One element is the equilibrium budget allocation rule, which is related to the literature on Colonel Blotto games (Gross and Wagner 1950; Shubik 1970; Myerson 1993; Roberson 2006).[14] The second element is the endogenous choice of the vote-buying budget that is reminiscent of the equilibrium outcome in all-pay contests (Hillman and Riley 1989; Baye, Kovenock, de Vries 1996; Siegel 2009). The main insight that allows for the proof of Theorem 1 is that the overall problem can be decomposed into these two elements: a budget choice problem and a budget allocation problem.

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[14]For the case where all voters are buyable, i.e., $\Delta = 0$, our budget allocation rules for fixed $a = b$ reduce to a structure that has been studied by Myerson (1993, Theorem 1).
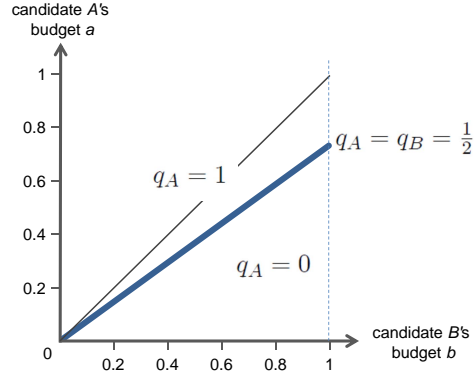
Figure 1: The bold line is $a(b)$ for $\Delta = 0.2$. Candidate A wins a majority of votes for budget combinations (a,b) above the bold solid line and loses the elections for budget combinations below this line.

The key to the intuition for the proof is to identify the sharp discontinuity in the equilibrium winning probability that emerges in a majoritarian system for all budget combinations $(a(b), b)$. This locus is mapped in Figure 1 for $\Delta = 0.2$.

If for two given budgets a and b, the budget of candidate A exceeds $a(b)$, then the equilibrium allocation rules lead to $\sigma_A > \frac{1}{2}$. The winning vote share may be far below 100 percent, but in a majoritarian system candidate A's winning probability q_A becomes equal to 1 for all combinations (a,b) above the thick line $a(b)$ in Figure 1. On the other side of the hyperplane $a(b)$, for $a < a(b)$, the equilibrium allocation rules imply that A's vote share is below 50 percent. In a majoritarian system this is as bad as a vote share of zero, as it implies that A will lose with probability 1. A crucial aspect of the proof of Theorem 1 is to show that unilateral deviations from the

equilibrium allocation rules do not affect this outcome at points which are not on the hyperplane defined by $a = a(b)$. It is the relative size of the budget which ultimately determines whether a player wins or not. It is this property that translates the problem into an all-pay contest in budget size.

One may argue that F_a and G_b as described in Theorem 1 may not matter much, simply because the set of combinations of (a, b) along $a = a(b)$ receives zero probability mass given the equilibrium choices Φ and Γ. However, the choices Φ and Γ interact with the choices of F_a and G_b and the candidates choose their budgets optimally as in the all-pay contest only in view of the budget allocation rules they and their competitor choose.

The equilibrium that is characterized in Theorem 1 has a number of interesting properties.

When all voters are buyable, i.e., $\Delta = 0$ for all voters, the two candidates jointly fully dissipate the office rent in expectation. Each of them expends a budget equal to half the value of the prize of winning office, and each of them wins with a probability of $\frac{1}{2}$.

If candidate A has loyal voters (or, more generally speaking, a larger share of loyal voters than candidate B), this turns out to be an advantage, even though candidates cannot observe who these loyal voters are. Candidates expend money on such voters, but these voters will always vote in line with their loyalties, even if the amount received from the candidate they like is less than the amount received from B. This means that candidate A will even win some of the voters which received more money from B than from A. There are different ways in which having loyal voters may be advantageous: it could increase A's equilibrium winning probability, it could reduce the amount of campaign spending that is needed by candidate A to win, or it could affect both the winning probability and the vote-buying expenditure. Theorem 1 shows that the advantaged candidate benefits in terms of win-

ning probability. Candidate A will win with a higher probability, but both candidates will choose the same expected budget size. This equilibrium outcome is characterized by candidates with a stronger electoral base winning more often, and with a campaign budget that does not necessarily exceed the campaign budget of the underdog candidate.

Theorem 1 is important for an assessment of the large literature that looks at campaigning as an all-pay contest. A large share of this literature either considers a black-box voting mechanism in which budget choices turn into winning probabilities or it applies the assumption that candidates cannot target their campaign expenditure at single voters or single groups of voters. Theorem 1 shows that candidates have an incentive to use the option to target their expenditure at single voters. Even though the individual voter type (loyal or buyable) is not observable for the candidates, they choose to allocate their budget for payments from a random distribution, such that different voters receive different payments. Thus, the equilibrium analysis reveals that candidates have an incentive to "cultivate favored minorities" as in Myerson (1993): some voters are treated very well and others are treated very badly. Even though the budget allocation rule is not uniform and voters receive different payments in the equilibrium, the budget choices follow rules very similar to the rules in the studies that treat the voter group homogeneously: the budget choice follows a pattern that is equivalent to the equilibrium effort choices in an all-pay contest.

4 Discussion

4.1 Private information about loyalty and political campaigns

We will now discuss the implications of the privacy of information about loyalty on political campaigns. As follows from Theorem 1, if the loyalty status of individual voters cannot be observed by candidates, then loyal and buyable voters will receive the same payments in expectation. The proposition below shows the outcome if we change only one assumption in the game that has been considered in the main section: we assume that voters' loyalty is observable by candidates.

Proposition 1 *Under complete information about the loyalty of voters, only the non-loyal voters receive positive transfers in expectation. The equilibrium distributions, expected campaign budgets, expected payoffs in this equilibrium, and winning probabilities follow respectively equations (1), (2), (3), (4), (5), (6), (7).*

The proof of Proposition 1 relies on the same forces as the proof of Theorem 1. Intuitively, switching from incomplete to complete information about loyalty makes candidates able to target their offers more precisely to buyable voters only. However, the budget choice and budget allocation rule have the same shapes under the two informational environments. This comes from the fact that the intensity of the competition for buyable voters increases simultaneously since both candidates free up the resources initially spent on loyal voters, and none of the candidates has an informational advantage in the two cases.

If the loyalty status of a voter can be observed by candidates, only non-loyal voters are targeted and get an expected positive transfer in equilibrium.

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This highlights why voters have a strong interest in not revealing their loyalty status. Indeed, if an individual voter were to reveal his loyalty to any of the candidates he would lose a positive expected transfer without being able to affect the candidates' probability of winning since the electorate is large. Hence, under incomplete information for both candidates, voters would not deliberately reveal their loyalty to any of the candidates.

A direct implication from Proposition 1 is that privacy of information about loyalty makes the campaign spending more inclusive in the sense that all voters can expect to get positive attention from the two candidates without affecting the expected total budget or winning probabilities.

4.2   Supermajorities

Supermajorities are an important and puzzling phenomenon in majoritarian systems. If 50.1 percent of the votes is enough to win, if it is expensive to campaign and acquire votes, and if candidates can approach single voters or voter groups and treat different voters differently, why would the equilibrium outcome often be characterized by one party receiving far more than half of the votes? In the introduction we pointed to some of the literature that posed this puzzle and addressed this puzzle.

Our results provide a new answer to this question. We focus on the case where there are no loyal voters, i.e., $\Delta = 0$. This is the natural case to consider since a perfect symmetry of candidates and voters makes the observation of supermajorities in majoritarian elections and the implied asymmetry in the outcome an even more intriguing phenomenon. The following proposition shows the implications of Theorem 1 for the emergence of supermajorities.

Proposition 2 *Supermajorities (defined as vote shares exceeding 1/2) occur in the vote-buying equilibrium defined in Theorem 1. For $\Delta = 0$, the probability of the winner's share being greater than or equal to $\sigma \in (\frac{1}{2}, 1)$ is*

$$prob(max\{\sigma_A, \sigma_B\} \geq \sigma) = \frac{1}{2\sigma}. \tag{10}$$

Proof For $\Delta = 0$ we can disregard voters whose voting decision is unaffected by the vote-buying efforts and the law of large numbers allows us to approximate the vote shares for a given a and b directly. The vote share σ_A, for instance, is equal to the probability that a randomly picked given individual voter i gets a higher payment from A than from B. Using the equilibrium distributions, the actual vote shares are

$$\sigma_A(a, b) = \frac{a}{2b} \text{ and } \sigma_B = 1 - \sigma_A.$$

Consider the distributions Φ and Γ. In the equilibrium these are uniform distributions on the unit interval. Geometrically, all possible combinations of a and b are represented by a unit square. Uniform density applies to all combinations (a, b). The size of an area in this unit square measures the probability of a random draw (a, b) being located in this area. The probability of the actual voting share σ_A of A being at least equal to some given $\bar{\sigma}_A > \frac{1}{2}$ is equal to the probability that $\frac{a}{2b} \geq \bar{\sigma}_A$, or, equivalently, $b \leq \frac{a}{2\bar{\sigma}_A}$. In the unit square, for $\bar{\sigma}_A \geq \frac{1}{2}$, this is an area equal to

$$\frac{1}{2\bar{\sigma}_A} \frac{1}{2}.$$

This is the probability of A receiving a supermajority of at least $\bar{\sigma}_A$ in the equilibrium. As the problem is symmetric, this is also the equilibrium probability of B obtaining a supermajority of this size or higher. Summing up these probabilities yields (10) and completes the proof. ∎

Intuitively, the uncoordinated choices of a and b yield unequal budgets, and these translate into unequal vote shares. For $\Delta = 0$, we find the density of supermajorities of the winning party of a given size also from (10). The first derivative yields

$$-\frac{\partial(2(\frac{1}{2\sigma}\frac{1}{2}))}{\partial\sigma} = \frac{1}{2\sigma^2}$$

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for the range $\sigma \in (\frac{1}{2}, 1)$.

Our result offers a solution to the puzzle described by Groseclose and Snyder (1996, 2000) and Banks (2000), but for the case of simultaneous budget choices by the two candidates. Groseclose and Snyder also offered a possible solution, but their solution relied on the assumption that the two candidates move sequentially, i.e., for a situation in which the challenger can commit making his own expenditure choice only once the expenditure choices of the incumbent have been made.

5   Concluding remarks

Political competition has many dimensions. We focused on campaign expenditures that bribe or influence voters and are of an all-pay nature: they are expended prior to the election and cannot be recovered by a candidate, whether he wins or not. The candidates compete in a majoritarian election and simultaneously make both their budget choices and their choices as to how to allocate this budget. We also allow for some voters not being influenced by the bribes received when deciding whether to vote for a particular candidate. This loyalty is the voters' private information.

Our main results show how the budget choices and the budget allocations are intertwined with the privacy of information about the voters' loyalty. Whereas our political game differs from previous studies that consider a Stackelberg framework where one candidate can commit to making vote-buying decisions when the other candidate has made his choices, our equilibrium also predicts the existence of supermajorities in majoritarian elections.

Appendix

Proof of Proposition 1

We first show that both politicians bribe only the non-loyal voters. The loyal voters vote for politician A regardless of the possible vote-buying allocation. Any dollar spent on a loyal voter will be wasted since it has a budgetary cost without increasing the probability of winning votes for any politician. This implies that the voters bribed by the politicians are the non-loyal voters only.

The proof of equilibrium distributions and their implication for expected budgets, payoffs, and winning probabilities follows the same steps as the proof of Theorem 1. The reasoning, however, applies to the share $1 - \Delta$ of buyable voters. This share is deterministic and candidates know the identity of the loyal voters. To win, candidate A wants to gain a fraction $\tilde{\sigma}_A$ of the mass $1 - \Delta$ of buyable voters so that $\tilde{\sigma}_A(1 - \Delta) = \frac{1}{2} - \Delta$ or equivalently $\tilde{\sigma}_A = \frac{1-2\Delta}{2(1-\Delta)}$; candidate B wants to gain a fraction $\tilde{\sigma}_B$ of the $1 - \Delta$ voters so that $\tilde{\sigma}_B(1 - \Delta) = \frac{1}{2}$ or equivalently $\tilde{\sigma}_B = \frac{1}{2(1-\Delta)}$.

The structure of the proof follows steps (I) to (IV) as for Theorem 1. (I) We consider the optimality of G_b as in (2) for a given b and given that the budget a is drawn from a distribution $\Phi(a)$ and is allocated according to the rule (1). (II) We consider the optimality of (1) for a given a if candidate B chooses (G_b, Γ). (III). We draw conclusions about the candidates' payoffs as functions of (a, b). (IV) We use these results to confirm the optimality of the choices Φ and Γ. (V) We confirm that A and B expend the same budget in expectation, that A has a higher equilibrium payoff and a higher probability of winning than B if $\Delta > 0$.

Step (I): We show that G_b maximizes B's winning probability for a given b, for *(i)* $a = \frac{1-2\Delta}{1-\Delta}b$, *(ii)* $a > \frac{1-2\Delta}{1-\Delta}b$, and *(iii)* $a < \frac{1-2\Delta}{1-\Delta}b$.

(i) We start with the case $a = \frac{1-2\Delta}{1-\Delta}b$. Since $F_a(x) = 1$ for $x \geq 2a\frac{1-\Delta}{1-2\Delta}$, the allocation rule F_a induces a vote share for B among the $1 - \Delta$ buyable

voters that is equal to

$$\tilde{\sigma}_B = \int_0^{2a\frac{1-\Delta}{1-2\Delta}} F_a(x)\,dG_b(x) + \int_{2a\frac{1-\Delta}{1-2\Delta}}^{+\infty} dG_b(x).$$

Inserting (1) and using $dG_b(y) = 0$ for all $y > 2b = 2\,a\frac{1-\Delta}{1-2\Delta}$, we find

$$
\begin{aligned}
\tilde{\sigma}_B &= \int_0^{2b} \left(1 - \frac{1-2\Delta}{1-\Delta} + \frac{1-2\Delta}{1-\Delta}\frac{x}{2a\frac{1-\Delta}{1-2\Delta}}\right) dG_b(x) \\
&= 1 - \frac{1-2\Delta}{1-\Delta} + \frac{1-2\Delta}{1-\Delta}\frac{1}{2a\frac{1-\Delta}{1-2\Delta}} \int_0^{2b} x\,dG_b(x) \\
&= 1 - \frac{1-2\Delta}{1-\Delta} + \frac{1-2\Delta}{1-\Delta}\frac{b}{2a\frac{1-\Delta}{1-2\Delta}} = \frac{1}{2}\frac{1}{1-\Delta},
\end{aligned}
$$

where the equality in the third line follows from the budget constraint, and the last equality follows from $b = a\frac{1-\Delta}{1-2\Delta}$.

We now show that $\tilde{\sigma}_B = \frac{1}{2}\frac{1}{1-\Delta}$ is the maximum vote share among the buyable voters $\hat{\sigma}_B$ that can be reached for any feasible $\hat{G}_b$ with an expected budget $b = a\frac{1-\Delta}{1-2\Delta}$. Note that

$$
\begin{aligned}
\hat{\sigma}_B &= \int_0^{+\infty} F_a(x)\,d\hat{G}_b(x) \\
&\leq 1 - \frac{1-2\Delta}{1-\Delta} + \frac{1-2\Delta}{1-\Delta}\frac{1}{2a\frac{1-\Delta}{1-2\Delta}} \int_0^{+\infty} x\,d\hat{G}_b(x) = \frac{1}{2}\frac{1}{1-\Delta},
\end{aligned}
$$

where the inequality in the second line is strict whenever $x > 2a\frac{1-\Delta}{1-2\Delta}$ has positive mass, and the last equality follows from the budget constraint and from $b = a\frac{1-\Delta}{1-2\Delta}$.

Accordingly, G_b yields the maximum vote share that is equal to $\frac{1}{2}\frac{1}{1-\Delta}$ and leads to $q_B = \frac{1}{2}$.

(ii) Now we turn to the case $a > \frac{1-2\Delta}{1-\Delta}b$. We show that G_b is also optimal for this range of budgets of candidate A. Note that

$$\tilde{\sigma}_B = \int_0^{2a\frac{1-\Delta}{1-2\Delta}} F_a(x)\,dG_b(x) < \frac{1}{2}\frac{1}{1-\Delta},$$

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and $q_B = 0$. To show the optimality of G_b in this range it is sufficient to show that no other $\hat{G}_b(x)$ exists that yields $\hat{\sigma}_B \geq \frac{1}{2}\frac{1}{1-\Delta}$. Consider

$$
\begin{aligned}
\hat{\sigma}_B &= \int_0^{+\infty} F_a(x)\, d\hat{G}_b(x) \\
&\leq 1 - \frac{1-2\Delta}{1-\Delta} + \frac{1-2\Delta}{1-\Delta}\frac{1}{2a\frac{1-\Delta}{1-2\Delta}}\int_0^{+\infty} x\, d\hat{G}_b(x) < \frac{1}{2}\frac{1}{1-\Delta}\ ,
\end{aligned}
$$

where the inequality in the second line is strict whenever $x > 2a\frac{1-\Delta}{1-2\Delta}$ has positive mass, the last strict inequality follows from the budget constraint and holds for all $a\frac{1-\Delta}{1-2\Delta} > b$. We conclude that $q_B = 0$ for all $\hat{G}_b(x)$ with expected budget b for all $a > \frac{1-2\Delta}{1-\Delta}b$.

(iii) Now we turn to the case $a < \frac{1-2\Delta}{1-\Delta}b$. In this case, inserting (1) and (2) yields

$$
\begin{aligned}
\tilde{\sigma}_B &= \int_0^{2a\frac{1-\Delta}{1-2\Delta}} F_a(x)\, dG_b(x) + \int_{2a\frac{1-\Delta}{1-2\Delta}}^{+\infty} dG_b(x) \\
&= \int_0^{2a\frac{1-\Delta}{1-2\Delta}}\Big(1 - \frac{1-2\Delta}{1-\Delta} + \frac{1-2\Delta}{1-\Delta}\frac{x}{2a\frac{1-\Delta}{1-2\Delta}}\Big)\frac{1}{2b}dx + \int_{2a\frac{1-\Delta}{1-2\Delta}}^{2b}\frac{1}{2b}dx \\
&= \frac{1-2\Delta}{1-\Delta}\int_0^{2a\frac{1-\Delta}{1-2\Delta}}\Big(\frac{x}{2a\frac{1-\Delta}{1-2\Delta}} - 1\Big)\frac{1}{2b}dx + \frac{1}{2b}\int_0^{2b} dx \\
&= -\frac{a}{2b} + 1 \\
&> 1 - \frac{\frac{1-2\Delta}{1-\Delta}b}{2b} = \frac{1}{2}\frac{1}{1-\Delta}.
\end{aligned}
$$

The strict inequality in the last line holds because $a < \frac{1-2\Delta}{1-\Delta}b$. Now, $\tilde{\sigma}_B > \frac{1}{2}\frac{1}{1-\Delta}$ implies $q_B = 1$, and this is the maximum winning probability. It cannot be increased by a different $\hat{G}_b(x)$. This concludes the first part of the proof.

Step (II): We now turn to the optimality of the allocation rule (1) for a given a, given that b is drawn from a distribution $\Gamma(b)$ and is allocated according to the rule (2). We confirm this again in three steps, showing that

F_a is an optimal reply if $b = \frac{1-\Delta}{1-2\Delta}a$, if $b > \frac{1-\Delta}{1-2\Delta}a$ and if $b < \frac{1-\Delta}{1-2\Delta}a$. It is easy to calculate that, applying the equilibrium actions, (1) and (2) imply that

$$
\begin{aligned}
\tilde{\sigma}_A &> \frac{1}{2}\frac{1-2\Delta}{1-\Delta} \text{ if } b < \frac{1-\Delta}{1-2\Delta}a \text{ , implying } q_A = 1, \\
\tilde{\sigma}_A &= \frac{1}{2}\frac{1-2\Delta}{1-\Delta} \text{ if } b = \frac{1-\Delta}{1-2\Delta}a, \text{ implying } q_A = \frac{1}{2}, \text{ and} \\
\tilde{\sigma}_A &< \frac{1}{2}\frac{1-2\Delta}{1-\Delta} \text{ if } b > \frac{1-\Delta}{1-2\Delta}a, \text{ implying } q_A = 0.
\end{aligned}
$$

We now treat $b = \frac{1-\Delta}{1-2\Delta}a$, if $b > \frac{1-\Delta}{1-2\Delta}a$ and $b < \frac{1-\Delta}{1-2\Delta}a$ as separate cases and consider whether there is an $\hat{F}_a(x)$ that yields a higher winning probability.

(i) For $b < \frac{1-\Delta}{1-2\Delta}a$, an $\hat{F}_a(x)$ could not increase the winning probability further, as it is already equal to 1.

(ii) For $b = \frac{1-\Delta}{1-2\Delta}a$ the allocation rule (1) induces a vote share for A that is equal to $\frac{1}{2}\frac{1-2\Delta}{1-\Delta}$ among the buyable voters and a winning probability $q_A = \frac{1}{2}$. We ask if there is an $\hat{F}_a(x)$ that yields a vote share that exceeds $\frac{1}{2}\frac{1-2\Delta}{1-\Delta}$. The vote share is

$$
\begin{aligned}
\hat{\sigma}_A &= \int_0^{+\infty} G_b(x) d\hat{F}_a(x) \\
&\leq \int_0^{+\infty} \frac{x}{2b} d\hat{F}_a(x) \\
&= \frac{1}{2}\frac{1-2\Delta}{1-\Delta},
\end{aligned}
$$

for all possible $\hat{F}_a$.

(iii) For $b > \frac{1-\Delta}{1-2\Delta}a$ the allocation rule (1) induces a vote share for A that is smaller than $\frac{1}{2}\frac{1-2\Delta}{1-\Delta}$ among the buyable voters and a winning probability $q_A = 0$. We ask again if there is an $\hat{F}_a(x)$ that yields a positive winning

probability. The vote share is

$$
\begin{aligned}
\hat{\sigma}_A &= \int_0^{+\infty} G_b(x) d\hat{F}_a(x) \\
&\leq \int_0^{+\infty} \frac{x}{2b} \, d\hat{F}_a(x) \\
&= \frac{a}{2b} \\
&< \frac{a}{2\frac{1-\Delta}{1-2\Delta}a} = \frac{1}{2}\frac{1-2\Delta}{1-\Delta}.
\end{aligned}
$$

The last strict inequality comes from $b > \frac{1-\Delta}{1-2\Delta}a$. This completes Step (II).

Step (III): The proof in (I) and (II) identified a hyperplane

$$
a(b) = \frac{1-2\Delta}{1-\Delta}b,
$$

which separates the $a-b$ space such that

$$
q_A(a,b) = \begin{cases} 1 & \text{if } a > a(b) \\ \frac{1}{2} & \text{if } a = a(b) \\ 0 & \text{if } a < a(b) \end{cases} ,
$$

and $q_B = 1 - q_A$.

Steps (IV) and (V) are identical to the ones in the proof of Theorem 1.

References

[1] Banks, Jeffrey S., 2000, Buying supermajorities in finite legislatures, American Political Science Review 94(3), 677-681.

[2] Barelli, Paulo, Srihari Govindan, and Robert Wilson, 2014, Competition for a majority, Econometrica 82(1), 271-314.

[3] Basinger, S.J., and H. Lavine, 2005, Ambivalence, information, and electoral choice, American Political Science Review 99(2), 169-184.

[4] Baye, Michael R., Dan Kovenock and Casper G. de Vries, 1996, The all-pay auction with complete information, Economic Theory 8, 291-305.

[5] Bierbrauer, Felix J., and Pierre C. Boyer, 2014, Efficiency, welfare, and political competition, CESifo Working Paper No. 4814.

[6] Che, Yeon-Koo, and Ian Gale, 1998, Caps on political lobbying, American Economic Review 88(3), 643-651.

[7] Crutzen, Benoit S.Y., and Nicolas Sahuguet, 2009, Redistributive politics with distortionary taxation, Journal of Economic Theory 144(1), 264-279.

[8] Dekel, Eddie, Matthew O. Jackson, and Asher Wolinsky, 2008, Vote buying: General elections, Journal of Political Economy 116(2), 351-380.

[9] Dekel, Eddie, Matthew O. Jackson, and Asher Wolinsky, 2009, Vote buying: Legislatures and lobbying, Quarterly Journal of Political Science 4(2), 103-128.

[10] Diermeier, Daniel, and Roger B. Myerson, 1999, Bicameralism and its consequences for the internal organization of legislatures, American Economic Review 89(5), 1182-1196.

[11] Erikson, Robert S. and Thomas R. Palfrey, 2000, Equilibria in campaign spending games: Theory and data, American Political Science Review 94(3), 595-609.

[12] Groseclose, Tim, and James M. Snyder, Jr., 1996, Buying supermajorities, American Political Science Review 90(2), 303-315.

[13] Groseclose, Tim, and James M. Snyder, Jr., 2000, Vote buying, super-majorities, and flooded coalitions, American Political Science Review 94(3), 683-684.

[14] Gross, Oliver Alfred, and R.A. Wagner, 1950, A continuous Colonel Blotto game, Research Memorandum RM-408, RAND Corporation, Santa Monica.

[15] Hersh, Eitan D., and Brian F. Schaffner, 2013, Targeted campaign appeals and the value of ambiguity, Journal of Politics 75(1), 520-534.

[16] Hillman, Arye L., and John G. Riley, 1989, Politically contestable rents and transfers, Economics and Politics 1(1), 17-39.

[17] Jamieson, Kathleen Hall, 2013, Messages, micro-targeting, and new media technologies, The Forum 11(3), 429-435.

[18] Jaryczower, Matias and Andrea Mattozzi, 2013, On the nature of competition in alternative electoral systems, Journal of Politics 75(3), 743-756.

[19] Kenski, Kate, Bruce W. Hardy, and Kathleen Hall Jamieson, 2010, The Obama victory: How media, money and message shaped the 2008 election, Oxford University Press, New York.

[20] Konrad, Kai A., 2002, Investment in the absence of property rights, European Economic Review 46, 1521-1537.

[21] Kovenock, Dan, and Brian Roberson, 2008, Electoral poaching and party identification, Journal of Theoretical Politics 20(3), 275-302.

[22] Kovenock, Dan, and Brian Roberson, 2009, Inefficient redistribution and inefficient redistributive politics, Public Choice 139(3-4), 263-272.

[23] Kovenock, Dan, and Brian Roberson, 2012, Conflicts with multiple battlefields, in: Michelle R. Garfinkel and Stergios Skaperdas (eds.). Ox-

ford Handbook of the Economics of Peace and Conflict, Oxford University Press, New York, 503-531.

[24] Lizzeri, Alessandro, and Nicola Persico, 2001, The provision of public goods under alternative electoral incentives, American Economic Review 91(1), 225-239.

[25] Meirowitz, Adam, 2008, Electoral contests, incumbency advantages, and campaign finance, Journal of Politics 70(3), 681-699.

[26] Myerson, Roger B., 1993, Incentives to cultivate favored minorities under alternative electoral systems, American Political Science Review 87(4), 856-869.

[27] Pastine, Ivan, and Tuvana Pastine, 2012, Incumbency advantage and political campaign spending limits, Journal of Public Economics 96(1), 20-32.

[28] Ridout, Travis N., Franz, Michael, Goldstein, Kenneth M. and William J. Feltus, 2012, Separation by Television Program: Understanding the Targeting of Political Advertising in Presidential Elections, Political Communication 29(1), 1-23.

[29] Roberson, Brian, 2006, The Colonel Blotto game, Economic Theory 29(1), 1-24.

[30] Sahuguet, Nicolas, and Nicola Persico, 2006, Campaign spending regulation in a model of redistributive politics, Economic Theory 28(1), 95-124.

[31] Shubik, Martin, 1970, Voting, or a price system in a competitive market structure, American Political Science Review 64(1), 179-181.

[32] Siegel, Ron, 2009, All-pay contests, Econometrica 77(1), 71-92.

[33] Skaperdas, Stergios, and Bernard Grofman, 1995, Modeling negative campaigning, American Political Science Review 89(1), 49-61.

[34] Snyder, James, 1989, Election goals and the allocation of campaign resources, Econometrica 57(3), 637-660.