A Competitive Idea-Based Growth Model with Shrinking Workers' Income Share

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Abstract

In this paper we present a model in which endogenous growth arises in competitive markets. Knowledge is described as a labor-augmenting factor used directly in the final goods' production. Firms demand both basic nonrival knowledge contents, which are supplied jointly and inelastically with raw labor, and further contents supplied by patent holders. This fact, together with Lindahl prices for knowledge, allows competition to work, while it also implies that workers' income share declines overtime. In a first version of the model with constant cost of knowledge production the first best is attained. In further versions of the model, in which the cost of knowledge production is allowed to change over time and externalities arise, in a decentralized economy a second best equilibrium occurs in the transitional period, while in the long run there is convergence to efficiency. As the asymptotic equilibrium exhibits strong scale effects, we propose a final version of the model with only weak scale effects under the assumption that combining labor and knowledge becomes increasingly difficult.

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Socrates: There are beds and tables in the world—plenty of them, are there not?
Glaucon: Yes.
Socrates: But there are only two ideas or forms of them—one the idea of a bed, the other of a table.

—Plato, The Republic, Book X, dialogue between Socrates and Glaucon, 380 BC

1 Introduction

The mainstream literature on endogenous growth maintains that a non-competitive market must occur somewhere in the economy, in order to provide private economic incentives for research.[1] In his classical paper, Romer (1990) presents a model in which each inventor, by patenting her blueprint, can become the sole producer of a differentiated capital or intermediate good, thus enjoying market power and receiving a monopoly profit that covers her research costs. Moreover, in his model research activity produces positive externalities, since the number of blueprint varieties operates as a multiplier of final goods' production. Hence, markets are incomplete.

Hellwig and Irmen (2001) and Boldrin and Levine (2008) are among the notable exceptions to the mainstream paradigm. Hellwig and Irmen (2001) show that atomless competitive markets in the production sector can finance the fixed cost of research out of inframarginal rents, an opportunity allowed by strictly increasing marginal costs at the level of each individual firm. In their model, the returns from innovation are privately appropriable only in the introductory period. Information about new findings spills over afterwards, thus paving the way for further innovation in all the firms, with increasing returns at the aggregate level and economic growth. The presence of externalities implies, however, that the first best is not achieved. Boldrin and Levine (2008), instead, depart from the assumption of ideas as nonrival goods by noting that the means of access to knowledge (such as the template of a new good) are rival and excludable. Moreover, replicating the first copy is time-consuming, and this fact, coupled with the impatience of consumers, implies that in a competitive market a scarcity rent will be paid by buyers. The inventor can thus sell the first copy at a price—which depends on the value of the services it delivers in terms of consumption and of replicability into further copies— earning a rent that may be sufficiently large to finance the indivisible cost born to produce the innovation. Whenever research activity can be financed this way, economic growth becomes viable even if the inventor is not granted a patent and thus has no IPR over further copies.

The motivation for this paper is to further pursue the challenging notion of 'competitive innovation', in a framework in which, however, the classical view about the nonrival nature of knowledge is maintained. In fact, we aim at providing a parsimonious set of assumptions

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[1]The idea that market power is necessary for inducing costly research dates back to Schumpeter (1911). It was brought up again in the '90s by many papers (besides Romer [1990], see, *e.g.*, also Grossman an Helpamn [1991], Peretto [1996] and Aghion and Howitt [1998]) and subsequently it has become a widely shared tenet in the literature on endogenous economic growth.

sufficient at supporting the viability of competition in the whole economy, that is, price taking behavior and zero profits at all levels, as well as market incentives for R&D activities and thus sustained economic growth. While the assumptions we arrive at are demanding, they capture some features that might prevail in the economy thanks to the characteristics that technological change assumed in the last decades. The first feature we consider is that technical progress leads more and more to the provision of immaterial goods such as computer programs, internet applications, business models, etc., which are patentable[2] and directly usable in the final goods production. It thus seems appropriate to assume that knowledge can also enter directly into the final goods production function, without always having to be incorporated into labor or other inputs. This direct penetration of immaterial contents into final goods production also implies that much more information than in the past is available to asses the value of the marginal product of knowledge (Grey and Grimaud [2015]).

The second feature that we consider is the "routinization" process (Acemoglu and Autor [2011]), which implies the possibility of codifying and automating many tasks—including cognitive ones—previously performed by workers. Bringing this trend to its limit, this substitution process in the medium-upper tier of the labor market would spare only creative tasks, which imply the elaboration of new models and ideas. But then the suppliers of such ideas need no more to be employees. Entrepreneurs who operate under the protection of laws on Intellectual Property Rights (IPR) are well suited to supply knowledge contents. One can thus expect that patented knowledge has the potential for widely displacing both medium and high-skilled workers. While on the other hand robotization has in principle the potential for substituting also the remaining tasks, *i.e.*, those of low-skilled workers, the perspective diffusion and economic impact of robots is still questioned (Gordon [2014]). The problem is mainly represented by the difficulty of routinization of tasks which require a level of adaptability and responsiveness which is nowadays difficult to achieve. Hence, at least for a while the process of substitution should spare low-skilled workers, endowed with the basic levels of knowledge, which are still needed to exploit patented intellectual contents.

In our model knowledge is a nonrival but excludable good, supplied both by workers and by patent holders. As knowledge contents coming from these two sources are perfect substitutes when used in the production of final goods, they are treated as just one factor, *i.e.*, knowledge. Moreover, we assume that workers supply just basic knowledge, representing the given common cultural endowment, jointly with raw labor, while knowledge advances are supplied only by patent holders.[3] Because workers are paid according to their knowledge supply, technically speaking a labor market is absent.[4] Labor is nevertheless fully and efficiently allocated to each

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[2]This trend for the U.S. can be dated back to 1998, when in the so called State Street Bank case a business method was declared patentable. Many other similar rulings followed with respect to software. For patentability in general see Eckert and Langinier (2013).

[3]By a continuity argument, the knowledge endowment of workers at the initial instant of time provides a marginal benefit and commands a price equal to that of the further contents produced under patents' protection.

[4]The lack of a labor market occurs, e.g., in one-sector AK models in which human and phisical capital are considered as perfect substitutes. Note also that most of the mainstream literature on neoclassical economic

firm thanks to its inelastic supply joint in fixed proportions with the given initial knowledge endowment.

A technical ingredient that allows us to render competition viable is represented by the Lindahl prices paid by the final goods' sector to compensate the suppliers of knowledge, which is a nonrival but excludable good. Lindahl prices ensure the "internalization" of externalities and support the exact equivalent of a competitive market equilibrium. While equilibria based on Lindahl prices are often deemed unrealistic, in our model they are viable because the problem of "revelation" of demand prices does not arise, thanks to the easily available information on firms' demand for knowledge. While in the basic version of the model knowledge is fully excludable, so that the first best is reached, in a further version externalities arise and a second best result is attained in the transitional period. Even in the latter case, however, in the long run the decentralized solution converges to the optimal Asymptotic Balanced Growth Path (ABGP).

As for knowledge use, following an approach similar to ours, also Chantrel et al. (2012) assume that disembodied knowledge is being directly used in the production of final goods. Although starting from different assumptions, they show as well that competition in the market for knowledge is sustainable under perfect excludability and with Lindahl prices, so that the distortion caused by knowledge spillovers vanishes. However, Chantrel et al. (2012) assume that the final goods' production occurs under increasing returns, so that competition is not viable in that market.

To describe the final goods' production, we use instead a general neoclassical production function with constant returns to scale. Weitzman (1998) stresses that assuming constant returns to scale in long term growth models is reasonable, as the role of capital and knowledge, universally recognized as the engines of growth, is that of "reducing the drag on growth from fixed background factors" [Weitzman (1998), p.344], such as labor. Decreasing returns specifically to knowledge can be the result of a rescaling of knowledge to describe the effective augmentation that it provides, keeping into account the problems arising when ideas are transferred from the laboratory into production, in terms of frictions, partial substitution of previously used results, etc. Increasing costs of this transfer translate into a "decreasing returns-like" situation. Hence, in our framework on one hand Lindahl prices fix the drawbacks stemming from the nonrivalry of knowledge, while, on the other hand, constant returns to scale support competition.

All the idea-based competitive growth models mentioned above routinely assume that the

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growth does not grant the labor market much more dignity, since, thanks to the assumption of labor being inelastically supplied, it turns out to be a residual element of the general equilibrium, in which the (competitive) equilibrium wage emerges as a value having the only role of clearing the market, a function that in our model is indirectly performed by the knowledge price. The minor role of the labor market is already evident in the original Ramsey (1928) Cass-Koopmans (1965) model, as well as in the Solow-Swan (1956) model (see Acemoglu [2009], pp. 30–31). Even when the assumption of a fixed labor supply is relaxed, as in the optimal taxation literature, it turns out that deviations of the wage from the labor marginal product caused by taxation negatively affect the economy at each instant, and thus are relatively less harmful for growth than intertemporal distortions directly caused by capital income taxation (see, *e.g.*, Chapter 24 in Hindriks and Myles [2013]).

R&D activity is not protected by patents, so that the inventor cannot unduly restrict the exploitation of her invention by others. In fact, in Hellwig and Irmen (2001) a patent would allow the inventor/producer of final goods to permanently prevent the spillover of the productivity increases deriving from her invention and thus harm economic growth; in Boldrin and Levine (2008) knowledge does not exist outside the private and excludable goods in which it is embodied, so that granting a patent would simply allow the holder a distortionary monopolistic right, not needed to finance the initial cost of invention. In our model, instead, the patent is at the same time useful and harmless. It is useful because it gives the patent holder standard full property rights, so that the free exploitation of research results—as happens in the Hellwig and Irmen model after the information has leaked—cannot occur. Patents are harmless because the possibility of restricting the embedding rival good (*e.g.*, the copies, as in Boldrin and Levine) does not arise because knowledge is disembedded. Hence the property right granted by the patent just plays the role of allowing the holders of the stock of knowledge to receive the Lindahl price, which is equal to the social marginal benefit. Of course, this is not to deny that in practice various forms of anticompetitive behavior involving patents might occur; we just aim at assessing the logical viability of a competitive system with patents.

The relevance of our results relies on the provision of a theoretical background for explaining the decline of the income share of labor at the advantage of the share going to intangibles and particularly to holders of patents, a stylized fact that has attracted much attention (Corrado et al. [2009], Karabarbounis and Neiman [2014], Koh et al. [2015]) and whose motivations are widely debated. The novel contribution of the paper is actually that of providing a rationale for a declining workers' income share in a growing competitive economy. This result arises because, as economic growth proceeds, the share of knowledge owned by patent holders increases, while the share owned by workers (the basic common cultural endowment) stagnates. Hence, while we do not address the problem of public policies, our results suggest that governments should focus primarily on redistributive interventions to improve social welfare. While efficient competitive markets are viable in the long run, policies aimed at correcting market failures should be considered in specific cases only in the transitional period.

The paper is organized as follows: after presenting the basic model in sections 2 and 3, in section 4 we consider the case in which the cost of knowledge production is not constant overtime, establishing in Section 5 that the decentralized equilibrium attains Pareto optimality only in the long-run. Section 6 provides a parameterized example, while a version of the model with population growth and weak scale effects is presented in section 7. Conclusions follow in section 8 and all mathematical proofs are postponed in the Appendix.

2   A Simple "Immaterial" Model

Our model is a Ramsey-type model of growth with endogenous creation of knowledge. The economy is composed of households, firms and the government. Households receive compensa-

tions for supplying inputs to the production sector, purchase a composite consumption good, which also represents the numeraire, and choose how much to save in order to accumulate new knowledge. There are two types of firms: one performing knowledge creation activities (R&D-firms) and one producing the final consumption good (F-firms).

2.1 Households

For now we shed population growth and assume that the size of the economy—*i.e.*, the total number of households—is constant. We adopt the standard assumption that all households have the same rate of time preference, $\rho > 0$, and an identical increasing and concave instantaneous utility function.

A. 1 *The aggregate representative consumer is endowed with an instantaneous objective $u\left(C\right)$, where C is aggregate consumption, with $u'\left(C\right) > 0$ and $u''\left(C\right) < 0$.*

Households' goal consists of choosing consumption in order to maximize their own lifetime discounted utility subject to the usual asset accumulation constraint, and their initial common knowledge endowment $A\left(0\right) = A_0 > 0$. Knowledge is assumed to be a nonrival but excludable good. The initial knowledge endowment A_0 belongs to the households in the sense that they share it as a part of their cultural heritage. Households supply A_0 to the final good producers at all instants $t \geq 0$ jointly with constant labor[5] $L\left(t\right) \equiv L$, and firms can exploit A_0 only by hiring workers. Thanks to such joint supply and the fact that labor supply is inelastic, workers are willing to supply labor even if they are compensated only as owners of their initial knowledge endowment A_0 whenever such compensation is strictly positive, each one being entitled to a share A_0/L of such compensation.

As the representative household earns only royalties from renting knowledge to F-firms while no wage is being earned for labor supplied, she faces the following maximization problem,

$$\max_{[C(t)]_{t=0}^{\infty}} \int_0^{+\infty} u\left[C\left(t\right)\right] e^{-\rho t}\, \mathrm{dt} \tag{1}$$
$$\text{subject to } \dot{B}\left(t\right) = r\left(t\right) B\left(t\right) - C\left(t\right),$$

where $B\left(t\right)$ denotes an asset that will be specified later on and $r\left(t\right)$ is the market rate of return on assets, with the additional constraint $0 \leq C\left(t\right) \leq r\left(t\right) B\left(t\right)$, for a given initial asset level $B\left(0\right) = B_0 > 0$. Standard analysis of the (concave) current-value Hamiltonian associated to (1) yields the following necessary condition of optimality, which is the well-known Euler equation:

$$\frac{\dot{C}\left(t\right)}{C\left(t\right)} = \frac{1}{\varepsilon_u\left[C\left(t\right)\right]}\left[r\left(t\right) - \rho\right], \tag{2}$$

where $1/\varepsilon_u\left(C\right) = -u'\left(C\right)/\left[u''\left(C\right) C\right]$ is the intertemporal elasticity of substitution.

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[5]For the sake of simplicity it is assumed that labor L is a continuous variable.

2.2 R&D Sector

Throughout the whole paper we will resort to the standard simplified approach that describes aggregate knowledge creation as a deterministic process. That is, we assume that there is no aggregate uncertainty in the innovation process while, of course, there may be idiosyncratic uncertainty.[6] In this first version of the model we simplify things further by considering a constant cost of new knowledge production, an assumption that will be relaxed later on.

A. 2 *Each new idea can be produced at a constant unit cost, $\eta(t) \equiv \eta > 0$.*

Under Assumption 2 the innovation possibilities frontier is given by

$$\dot{A} = \frac{J}{\eta}, \tag{3}$$

where J represents investment in new knowledge production. We shall assume that there is free entry into R&D activities, that is, any individual or firm can spend one unit of its wealth at time t to generate a flow rate $1/\eta$ of new ideas. Every $R\&D$-firm produces new knowledge and aims at profit maximization. In order to obtain an infinitely lived and fully enforced patent, knowledge must be new and differentiated, while, as will be clarified in the next section, when knowledge is used by the F-firms it behaves as a homogeneous good. Because the number of $R\&D$-firms and in general of knowledge providers is large, the market for each invention is competitive and each $R\&D$-firm makes zero profits. Hence, the value of the patent associated to each (differentiated) unit of new knowledge purchased by households at instant t corresponds to the same constant: $\eta > 0$.

2.3 Producing Sector

In the final good sector F-firms are competitive and operate in a standard neoclassical framework: at each instant t F-firm i employs a composite intermediate good and knowledge-augmented labor to produce a composite consumption good according to a neoclassical production function, $Y_i = F(X_i, A_i L_i)$. The intermediate good X is made up of final goods destined to production, so that its price is the numeraire, $p^X = 1$, while L is labor provided by a large and constant labor population and A denotes knowledge. Recall that knowledge is assumed to be nonrival and perfectly excludable.[7] Specifically, from the F-firms perspective A is an aggregate composition of perfect substitutes, implying that whichever new idea is added to the stock, it has the same marginal productivity of all other ideas—including those forming the initially inherited endowment A_0. Knowledge can be directly used in production to augment the effectiveness of labor. Because labor is supplied inelastically and jointly with the initial knowledge A_0, while all other inputs are for sale, F-firms demand only the intermediate goods

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[6]See, *e.g.*, pp. 428–429 in Acemoglu (2009).

[7]F-firms cannot re-rent knowledge or share it with other firms; that is, no arbitrage is allowed.

X and knowledge A. Input X is bought on the market, while knowledge is rented from a large set of suppliers, *i.e.*, both from workers—who own the initial endowment A_0—and from patent holders—who own the patents on knowledge supplied by $R\&D$-firms after the initial instant $t = 0$. As will become clearer in the following, the amount of labor needed to produce each firm's output amount, Y_i, is being supplied for free according to the firm's knowledge demand.

A. 3 $F(\cdot, \cdot)$ *is concave and linearly homogenous, with $F_1 > 0$, $F_2 > 0$, $F_{11} < 0$ and $F_{22} < 0$, where F_j and F_{jj} denote the first-order and second-order partial derivatives with respect to arguments $j = 1, 2$ respectively. Moreover, the standard Inada conditions hold for both arguments.*

Consistently to the fact that each firm's maximization problem has only two decision variables, X and A, Assumption 3 rules out increasing returns to scale at the level of the single firm. Assumption 3 also postulates decreasing returns to knowledge for the production function F. This can be rationalized by considering that the transfer of knowledge from the research sector to the final good's production involves a rescaling in order to take into account its decreasing effectiveness in terms of output augmentation.[8]

Because households own the whole knowledge stock A—the initial endowment A_0 plus the amount $(A - A_0)$ purchased from $R\&D$-firms—they are willing to supply the whole lot provided that they can earn a strictly positive price on it. Specifically, denoting by A the aggregate demand for knowledge, by A^* the whole endowment available in the economy and by p^A the rental price of knowledge, the market clearing conditions for knowledge can be written in complementary slackness form as

$$A \leq A^*, \ p^A \geq 0 \qquad \text{and} \qquad (A^* - A)\, p^A = 0,$$

which imply $A = A^*$ whenever $p^A > 0$, as will be the case in the following. Indeed, in the instantaneous market equilibrium knowledge can be considered in fixed supply, as it is a stock variable:[9] hence, at each instant, the equilibrium price p^A depends only on demand. However, knowledge is a nonrival good; therefore, all F-firms can be supplied at the same time with the whole stock A^* available, and the subscript i can be dropped from the amount of knowledge employed in their production function.

Let us reformulate firm i output as

$$Y_i = A L_i f\left(\frac{x}{A}\right), \qquad \text{with } f(\cdot) = F(\cdot, 1), \tag{4}$$

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[8]A decreasing marginal product of knowledge may result when one takes into account an accelerating depreciation of the pre-existing stock, due, *e.g.*, to partial substitution of previously used results with new knowledge.

[9]The intertemporal market equilibrium in which the A^* values available at different instants are determined will be analyzed in section 3.

where[10] $x = X_i/L_i$ is the per capita intermediate good. As Assumption 3 envisages linear homogeneity, the scale and the number of firms are indeterminate, so that we can assume without loss of generality that at each instant t there is a large number of firms, say $M(t)$, and several, say $N(t) < M(t)$, output levels $Y_i(t)$, for $i = 1, \ldots, N(t)$, each corresponding to a different amount L_i of labor employed, that are being produced by $m_i(t)$ identical firms operating at the same level, with $m_i(t) \geq \underline{m} > 0$ where $\underline{m}$ is a number sufficiently large to sustain a competitive market. Therefore, at each instant t the economy is populated by $M(t) = \sum_{i=1}^{N(t)} m_i(t)$ firms producing a total amount $Y(t) = \sum_{i=1}^{N(t)} m_i(t) Y_i(t)$ of final consumption good.[11] If $N(t) = 1$ then $M(t) = m_1(t)$ and $Y(t) = m_1(t) Y_1(t)$. Knowledge suppliers can observe the firms' size and thus tell apart each $m_i(t)$ group of firms.

The FOC for each F-firm's profit maximization are:

$$\frac{\partial Y_i}{\partial X_i} = f'\left(\frac{x}{A}\right) = 1 \tag{5}$$

$$\frac{\partial Y_i}{\partial A} = L_i\left[f\left(\frac{x}{A}\right) - \frac{x}{A}f'\left(\frac{x}{A}\right)\right] = L_i\gamma\left(\frac{x}{A}\right) = p_i^A. \tag{6}$$

Condition (5) holds because the intermediate goods are priced at the numeraire, while in condition (6) the term $\gamma(x/A)$ denotes the equilibrium royalty per augmented worker, which depends on the ratio x/A. Condition (6) shows that the equilibrium price for knowledge, p_i^A, is the firm i demand price evaluated at the given amount $A = A^*$.

As there are many F-firms operating at each output level Y_i, one can identify $N(t)$ independent and competitive sub-markets for knowledge. In each sub-market, the demand price is scaled by the labor amount L_i needed to produce its representative firm specific output size, Y_i, according to (4). Because A is nonrival and inelastically supplied, all the sub-markets clear at the same amount $A = A^*$. However, because F-firms demand prices are scaled according to L_i, the knowledge equilibrium rental price in each sub-market will differ accordingly; in other words, $p_i^A = L_i\gamma(x/A)$ are Lindahl prices.[12] Households who invested in new patented knowledge receive payments from all the firms in the economy and earn a share of total output, Y, given by $L\gamma(x/A)(A - A_0)/Y$, while, in their role of workers who rent A_0 to all F-firms, households receive $L\gamma(x/A)A_0/Y$.

Example 1 *Assume that all firms employ the Cobb-Douglas production function $F(X_i, L_iA) = X_i^\alpha(L_iA)^{1-\alpha}$ with $0 < \alpha < 1$, that is, a firm size is given by $Y_i = X_i^\alpha(L_iA)^{1-\alpha} = L_iA(x/A)^\alpha$;*

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[10] All firms employ the same intermediate good/labor ratio, $x \equiv x_i = X_i/L_i$, as will become clear in the sequel.

[11] Alternatively, one can assume that there is a continuum of output levels $Y(i,t) \geq 0$, $i \in [0, N(t)]$, each produced by a density $m(i,t) \geq 0$ of identical F-firms, so that the economy is populated by an absolutely continuous distribution of firms over the compact support $[0, N(t)]$, and total output is given by $Y(t) = \int_0^{N(t)} m(i,t) Y(i,t)\,di$. Note that when $m(i,t) > 0$ there is a continuum of firms each producing $Y(i,t)$, thus assuring that such sub-market is competitive, while if $m(i,t) = 0$ there are no firms producing the $Y(i,t)$ level.

[12] A similar approach has been pursued in Chantrel et al. (2012) with reference to knowledge demanded by producers of differentiated goods, each one with its specific sub-market and Lindahl price. They also show that under perfect information and full excludability this is the only market equilibrium.

moreover, assume that at a given instant the inelastic supply of knowledge is A^*. Thus, the optimal per capita intermediate good x, labor L_i and the equilibrium Lindahl price $p_i^A = L_i \gamma (x/A)$ of firm i are the solution of the following system of equations:

$$\begin{cases} Y_i = L_i A^* (x/A^*)^\alpha \\ \alpha (x/A^*)^{\alpha-1} = 1 \\ L_i (1-\alpha) (x/A^*)^\alpha = p_i^A, \end{cases} \quad (7)$$

where the first equation simply recalls the (arbitrary) choice of firm i on its size, Y_i, which must satisfy (4), the second equation is condition (5) and the third equation is condition (6), all holding for A^* given. Then, the optimal amount of the intermediate good is obtained as $X_i = xL_i$. For example, if $\alpha = 0.33$, $Y_i = 100$ and $A^* = 10$, the second equation in (7) becomes $0.33 (x/10)^{-0.67} = 1$ which yields $x = 1.91$; using this value, the first equation in (7) becomes $100 = 10L_i (0.191)^{0.33}$, yielding $L_i = 17.26$, so that $X_i = xL_i = 33$. Finally, the last equation in (7) provides the Lindahl price firm i is willing to pay for the knowledge amount $A^* = 10$ used as $p_i^A = 6.7$, corresponding to a total expenditure for knowledge of $p_i^A A^* = 67$. Note that the values $p^X X_i = X_i = 33$ (where $p^X = 1$ because X_i is the numeraire) and $p_i^A A^* = 67$ satisfy $X_i + p_i^A A^* = 100 = Y_i$, consistently with the well-known property of a Cobb-Douglas production function according to which the exponents determine the factor share of each input in percent terms; note also that the whole income is distributed to the factors of production and there is no profit. The income distributed to knowledge is paid to the owners of this factor according to their entitlements. For instance, assuming that $A_0 = 2$, workers, who own A_0, receive $p_i^A A_0 = 13.4$, while the remaining income gained by knowledge, i.e., $p_i^A (A^* - A_0) = 67 - 13.4 = 53.6$ goes to patent holders.

Figure 1 shows how Lindahl prices form for two hypothetical sub-markets $i = 1, 2$. Without loss of generality, suppose that there is a continuum of firms in each sub-market, both with mass $m_1 = m_2 = 1$. While all the firms in each sub-market have the same size, sub-market 1 is populated by firms which are smaller in size than those operating in sub-market 2, that is, $Y_1 < Y_2$, so that, according to (4), $L_1 < L_2$. Lindahl prices $p_i^A = L_i \gamma (x/A^*)$, for $i = 1, 2$, are determined by the intersection of the marginal products of knowledge given by $\partial Y_i / \partial A = L_i \gamma (x/A)$—which, for fixed L_i and $x = X_i/L_i$, is the representative firm's demand for knowledge in the ith sub-market[13] as a function of the only variable A—with the amount A^* in fixed supply. According to Assumption 3, the demand functions $L_i \gamma (x/A)$ are decreasing in A when L_i and $x = X_i/L_i$ are kept constant. The upper black demand curve in Figure 1 refers to the whole knowledge market; it is obtained as the "vertical sum" of the sub-markets' demands, yielding the sum of Lindahl prices for knowledge in the whole economy for any given A. At the intersection point of the total market demand with the amount A^* inelastically supplied, the royalty for knowledge—that is, the Lindahl price of the whole lot A^*—is determined as $p^A = L\gamma (x/A^*)$,

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[13]As $m_1 = m_2 = 1$, in Figure 1 the representative firm's demand coincides with the total sub-market demand.
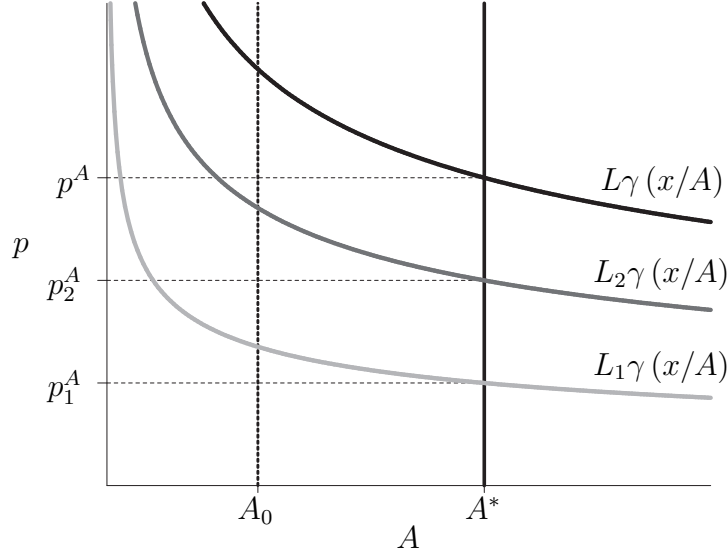
where $L = L_1 + L_2$.

FIGURE 1: equilibrium Lindahl prices in the two sub-markets, p_1^A, p_2^A, and in the whole knowledge market, p^A, as the intersection between sub-markets knowledge demand functions $L_1\gamma\,(x/A)$, $L_2\gamma\,(x/A)$, and the whole demand function $L\gamma\,(x/A)$, with the inelastic supply function $A \equiv A^*$.

In Figure 1 also the inelastic supply of the basic knowledge endowment, A_0, is plotted as a dashed vertical line to the left of the total supply A^*. Because the composite input $A_0 L$ is supplied in fixed amount, denoting by

$$h = A_0/L$$

the (constant) per capita initial knowledge endowment and after conveniently rescaling L through factor h, A_0 can alternatively be interpreted as the inelastic labor supply $A_0 = hL$. As labor is supplied for free jointly with A_0, when the knowledge market is in the equilibrium—i.e., when the whole A_0 is rented at the royalty $p^A = (L_1 + L_2)\,\gamma\,(x/A^*)$—also the total labor force is employed, while the allocation of labor to each firm is proportional to the firm's Lindahl price, that is,

$$\frac{p_i^A}{p^A} = \frac{L_i\gamma\,(x/A^*)}{(L_1 + L_2)\,\gamma\,(x/A^*)} = \frac{L_i}{L}.$$

While in general the resort to Lindahl prices for public goods is deemed unattainable due to the lack of proper information, in this case the tie between knowledge and labor allows to overcome the problem of demand revelation. The observable size and the corresponding labor input used by each firm render it possible to ascribe it to a specific sub-market, and all the suppliers of knowledge can easily identify such sub-markets and the demand arising therein. The demand revelation problem is in general less severe when the public good is an input than when it is a consumption good, as demand in the former case derives from the profit function while in the latter it derives from the utility function which has a lower degree of measurability

(Dasgupta [2001]).

According to (5) and (6) the per capita product $y_i = Y_i/L_i = Af(x/A)$ is fully distributed to the per capita intermediate good, $x = X_i/L_i$, and knowledge, A, that is,

$$y_i - x - \gamma(x/A)A = Af(x/A) - f'(x/A)x - Af(x/A) + xf'(x/A) = 0,$$

so that each firm (and thus the whole industry) makes no profit, as occurs in Example 1. Households, either in their role of workers as suppliers of A_0 or as holders of the patents covering $(A - A_0)$, are paid according to their entitlements. Because in the market for final goods all firms face the same rental price for intermediate goods and face Lindahl prices for knowledge, all use the same combination of intermediate goods and knowledge-augmented labor. We thus refer in the following to a representative firm and drop the subscript i.

The absence of a specific price for labor does not imply efficiency losses, because labor is inelastically supplied and efficiently allocated. One might also liken knowledge-augmented labor AL to human capital, but in our model the scale through which human capital is evaluated is provided by the embodied knowledge contents, which, notwithstanding such embodiment, are still homogeneous with respect to the dematerialized contents protected by patents. Because in our economy the dematerialization process is prevailing, only the basic human capital—embedding just the initial knowledge endowment A_0—is demanded. A noteworthy consequence is that workers are not paid their marginal product, that is, they do not receive the marginal product with respect to L. This possibility has already been considered in the literature; for instance, Hellwig and Irmen (2001) describe a competitive equilibrium in which, under inelastic labor supply, a decoupling between labor compensation and marginal labor productivity occurs. We take this decoupling to its extreme, as we aim at capturing, of course under strong simplifications, the overwhelming relevance of knowledge contents sold on the market in nowadays economies as a driver of economic growth.

One implication of the model is that the benefits of technical progress are funneled to the owners of the intangible capital protected by patents, while workers' compensation is stagnating as it is linked to the original knowledge endowment. The negative consequence of the model with respect to workers' income accords, at any rate, with the stylized facts pertaining to the long term evolution of the income shares. Corrado et al. (2009), *e.g.*, show that labor's income share decreased significantly over the last 50 years in the US, provided that investments in intangibles and their income share are properly accounted for. These negative effects might be mitigated if the role of worker and that of licensee overlap. As long as worker's abilities can increase in this way, firms can pay Lindahl prices to these workers-intermediaries. In this case the option for either a direct access (through patent renting) or an indirect one (through trained workers) would imply the consideration of the respective transaction costs, an extension of the model that we leave for future research.

2.4 Instantaneous Market Equilibrium

We end this section with a summary of the main features that characterize the equilibrium in the instantaneous, static knowledge market.

Proposition 1 *Under Assumptions 1–3 the competitive market equilibrium for knowledge at any instant $t \geq 0$ is attained at the following conditions:*

i) *labor L is inelastically supplied to F-firms by households joint with inherited basic knowledge A_0, workers receive a positive remuneration only because they carry the knowledge amount A_0 to the firms;*

ii) *the whole stock $A(t)$ is inelastically supplied to F-firms by households in their role of workers (the initial share A_0) and of patent holders (the share $[A(t) - A_0]$ accumulated and patented by R&D-firms at the constant cost $\eta > 0$ up to instant t) for a positive compensation in terms of royalties;*

iii) *there is a huge number $M(t)$ of F-firms distributed in $N(t) < M(t)$ sub-markets, in each of which a large number $m_i(t)$ of identical firms produce the same output amount $Y_i(t)$ and pay the same Lindahl price $p_i^A(t) = L_i \gamma [x(t)/A(t)]$ for the whole knowledge lot $A(t)$ supplied by households, for $i = 1, \ldots, N(t)$;*

iv) *different sub-markets are characterized by different equilibrium Lindahl prices $p_i^A(t) = L_i \gamma [x(t)/A(t)]$, each scaled according to the number of workers L_i employed by all F-firms in the same sub-market, while the total royalty paid to all knowledge suppliers in the economy corresponds to the (vertical) sum of all Lindahl prices: $p^A(t) = \sum_{i=1}^{N(t)} p_i^A(t) = L\gamma [x(t)/A(t)]$.*

3 Intertemporal Competitive Equilibrium

Definition 1 *A feasible allocation in this economy is a set of time paths of consumption levels $[C(t)]_{t=0}^{\infty}$, intermediate goods' flows $[X(t)]_{t=0}^{\infty}$, aggregate R&D expenditures $[J(t)]_{t=0}^{\infty}$, and knowledge stock levels $[A(t)]_{t=0}^{\infty}$, such that, at each instant $t \geq 0$, the aggregate resource constraint, $C(t) + X(t) + J(t) \leq F[X(t), A(t)L]$, is satisfied, while new knowledge production occurs according to $\dot{A}(t)\eta(t) \leq J(t)$, where $\eta(t)$ is the amount of final goods technically needed to produce 1 unit of $A(t)$.*

In order to incentivate all households to invest in new knowledge production, while keeping the knowledge market in intertemporal equilibrium, under Assumption 2 the following free-entry condition must hold:

$$V(t) = \int_t^{+\infty} \gamma(v) L e^{-\int_t^v r(s)\,\mathrm{d}s}\,\mathrm{dv} = \eta. \tag{8}$$

It postulates that the present value [$r(t)$ is the instantaneous interest rate] of future royalties, $\Gamma(t) = \gamma(t) L$, where $\gamma(t)$ denotes the equilibrium royalty per worker at instant t, must be equal to the (constant) production cost η of a unit of new knowledge. Because the only asset in the economy is expressed in terms of the knowledge stock A owned by households, a first consequence of (8) is that $B(t) = V(t) A(t) = \eta A(t)$ must hold.

Definition 2 *An equilibrium is a feasible allocation in which the evolution of $[A(t)]_{t=0}^{\infty}$ is determined by free entry in a competitive market for R&D, the evolution of $[r(t)]_{t=0}^{\infty}$ is consistent with the free entry property, the evolution of $[C(t)]_{t=0}^{\infty}$ is consistent with household maximization, both R&D-firms and F-firms maximize their profit, Lindahl prices $\left[p_i^A(t)\right]_{t=0}^{\infty}$ clear each sub-market for knowledge when all F-firms use the whole available stock A and the optimal number of workers L_i, the royalties $\left[p^A(t)\right]_{t=0}^{\infty}$ clear the whole market for knowledge with full employment of labor. In other words, in equilibrium at each instant $t \geq 0$, $J(t)$ is consistent with (8) while $[C(t), r(t)]$ satisfy the Euler equation (2) and the transversality condition $\lim_{t \to +\infty} B(t) e^{-rt} = \lim_{t \to +\infty} \eta A(t) e^{-rt} = 0$ holds.*

As, under Assumption 3, the derivative of function f defined in (4) is decreasing, f' is invertible and from (5) we get the demand for the intermediate good x by the F-firms, which turns out to be linear in A:

$$x = \delta A, \qquad \text{with } \delta = (f')^{-1}(1). \tag{9}$$

Thus, δ is a constant uniquely defined by the choice of the production function $F(\cdot, \cdot)$. As the ratio $\delta \equiv x/A$ is constant, from (6) we obtain the per capita willingness to pay for knowledge, which turns out to be constant as well:

$$\gamma\left(\frac{x}{A}\right) \equiv \gamma = f(\delta) - \delta f'(\delta) = f(\delta) - \delta, \tag{10}$$

where in the third equality we used the definition of δ in (9).

Hence, we can use the fact that, according to (10), γ is constant and differentiate both sides in (8) with respect to t to get

$$\dot{V}(t) = r(t) \int_t^{+\infty} \gamma L e^{-\int_t^v r(s)\,\mathrm{d}s}\,\mathrm{d}v - \gamma L = 0,$$

which implies that the present value of future royalties does not change in time and, after substituting the integral with (8), the equilibrium interest rate regulating the transfer of wealth through time (via the only state variable, which is the knowledge A) is constant as well, $r(t) \equiv r$, and given by

$$r = \frac{\gamma L}{\eta}. \tag{11}$$

In order to use (2) to look for a *balanced growth path* (BGP) type of equilibrium, we must assume that $1/\varepsilon_u\left(C\right) = 1/\sigma$ is a constant,[14] with $\sigma > 0$. Hence, using (11) in (2) we obtain the following constant rate of growth of consumption, C, knowledge, A, and output, Y, along the BGP:

$$g = \frac{\dot{C}}{C} = \frac{\dot{A}}{A} = \frac{\dot{Y}}{Y} = \frac{1}{\sigma}\left(\frac{\gamma L}{\eta} - \rho\right). \tag{12}$$

Proposition 2 *Suppose that Assumptions 1–3 hold and the intertemporal elasticity of substitution is constant—i.e., $\varepsilon_u\left(C\right) = -\left[u''\left(C\right)C\right]/u'\left(C\right) \equiv \sigma > 0$. Then, if*

$$\gamma L > \rho \eta \qquad and \qquad \left(1 - \sigma\right)\gamma L < \rho \eta, \tag{13}$$

the economy admits a unique BGP along which knowledge, output, and consumption all grow at the same rate $g > 0$ given by (12). Moreover, there are no transition dynamics: the economy immediately jumps on the BGP starting from $t = 0$.

The novelty introduced by assuming that F-firms, operating under constant returns to scale, pay for the use of knowledge through Lindahl prices both to compensate patent holders and suppliers of basic knowledge joint with labor, together with Assumption 2 of a constant cost for new knowledge production, allows for the solution characterized in Proposition 2 to be Pareto optimal even in a totally decentralized setting.

Proposition 3 *The BGP equilibrium characterized in Proposition 2 is Pareto optimal.*

Three features of our model are crucial to explain Proposition 3.

1. As production is constant returns to scale and knowledge is paid its Lindahl price there is no room for monopolistic power exploitation in the economy as in the standard literature, that is, all prices turn out to be competitive and nowhere mark-ups are being applied. Moreover, Lindahl prices imply that the externalities deriving from the public good nature of knowledge are internalized. Actually, one can read (8) as a dynamic version of the famous Samuelson rule for public goods, since it establishes the equality between the present value of the sum (the integral) of the marginal benefits of knowledge and the marginal cost of knowledge production. This implies that knowledge is efficiently provided, as the same choices that a social planner would have made with respect to knowledge accumulation emerge in a decentralized setting.

2. As labor supply is fixed and joint with initial knowledge supply, it does not command a specific compensation, and the GDP is exhausted by paying knowledge and the intermediate goods their marginal product. Moreover, there is no inefficiency in the use of labor.

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[14]If we are interested in an asymptotic balanced growth path (ABGP) it is sufficient to assume that $\lim_{t\to+\infty}\left[1/\varepsilon_u\left(C\right)\right] = \lim_{t\to+\infty}\left\{-u'\left(C\right)/\left[u''\left(C\right)C\right]\right\} = 1/\sigma$. This approach will be pursued in Section 4.

The assumption of a fixed labor supply is standard in problems of long term growth (see footnote 4). If in this model we assumed variable labor supply, while maintaining that of jointness of labor supply with that of A_0, at instant $t = 0$ firms would demand labor instead of knowledge, as the latter would be made available by workers for free. In this case the total income received by workers at $t = 0$ would still be $A_0 p^{A_0} = A_0 L \gamma (x/A_0)$, *i.e.*, the same as in Proposition 1(iv), because they would still be entitled to the income share that factor $A_0 L$ deserves in a competitive market.[15] Because no price of knowledge would arise, no further knowledge would be produced by a competitive market, so that the workers' income would not differ from that indicated so far. As we are interested in a growing economy, we did not consider this case. One can note, however, that as long as knowledge piles up the workers' compensations in our model fall short of what they would have received under elastic labor supply and with research publicly funded by a benevolent social planner. The stagnation of workers' income would be particularly problematic in an economy in which, unlike that described in this paper, the population were composed by two groups: workers (endowed only with labor L and A_0), who consume their all income, and holders of the remaining assets, who consume and save. While we leave for future research the study of this case, one can guess that in this economy an evolution of income shares according to the pattern described so far is likely to prompt protests against the raising income inequality and give rise to social pressure for redistributive policies.

3. Finally, Assumption 2, by postulating a constant unit cost η for the production of new ideas, rules out intertemporal knowledge spillovers or other types of externalities. Our next step is to relax this assumption.

4 Non Constant Cost of Knowledge Production

An example of a well-known model in which the cost of knowledge production turns out to be constant like in our simple setting discussed in the previous sections is the celebrated original contribution by Romer (1990). However, as most of the major contributions that followed Romer's seminal paper confirm, it is widely accepted that Assumption 2 introduces a definitely unrealistic restriction. For instance, Tsur and Zemel (2007) consider a continuous-time version of the original model by Weitzman (1998) in which knowledge evolves according to a recombinant technology characterized by a variable unit cost of knowledge production,

$$\eta (t) = \varphi [A (t)], \tag{14}$$

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[15]To see this, note that $\partial Y_i / \partial L_i = A [f (x/A) - (x/A) f' (x/A)] = A\gamma (x/A)$, so that only the scale in which the marginal product is measured changes from L_i to A. The effect of this change of scale is cancelled out, however, when the total compensation received by all workers is calculated, as $\sum_{i=1}^{N(t)} m_i (t) L_i A\gamma (x/A) = AL\gamma (x/A) = Ap^A$, which is the same as that predicted by Proposition 1(iv) also for the case $A = A_0$.

in which the unit cost function $\varphi(\cdot)$ indirectly depends on time through the knowledge stock $A(t)$ evolution. The recombinant knowledge production function by Weitzman is well suited for our approach because it assumes that its main input is expressed in terms of financial resources,[16] $J(t)$. Hence, while we keep the assumption of Section 2 that there is no aggregate uncertainty in the innovation process, in this section we allow the unit cost of knowledge production, η, to vary over time.

A. 4 *Each new idea can be produced at a time-dependent unit cost, $\eta(t) > 0$.*

According to Assumption 4, define the innovation possibilities frontier as

$$\dot{A}(t) = \frac{J(t)}{\eta(t)}, \tag{15}$$

in which time-dependence has been emphasized.[17]

From the representative household optimization problem (1) we get the same necessary Euler condition as in (2). Also, nothing changes from the point of view of knowledge demand: the per augmented worker royalty is still constant and, according to (10), given by $\gamma(x/A) \equiv \gamma = f(\delta) - \delta$. Only the free-entry condition (8) changes, as now the RHS depends on time:

$$V(t) = \int_t^{+\infty} \gamma(v) L e^{-\int_t^v r(s)\,\mathrm{d}s}\,\mathrm{d}v = \eta(t). \tag{16}$$

Differentiating it with respect to time leads to

$$\dot{V}(t) = r(t) \int_t^{+\infty} \gamma L e^{-\int_t^v r(s)\,\mathrm{d}s}\,\mathrm{d}v - \gamma L = \dot{\eta}(t),$$

which, after substituting the integral with (16), yields the interest rate

$$r(t) = \frac{\gamma L}{\eta(t)} + \frac{\dot{\eta}(t)}{\eta(t)}. \tag{17}$$

Note that, although the total royalty $\Gamma = \gamma L$ remains constant, under Assumption 4 the interest rate varies in time according to the law of motion of $\eta(t)$.

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[16] As a matter of fact, all mainstream extensions of Romer's model implicitly assume time-dependent costs of producing new ideas. However, like Romer's one, all these models are based on knowledge production functions that use labor as a main input factor, so that the cost of new knowledge depends on the equilibrium wage. Because, according to (3) and (15), in our setting we assume that knowledge is produced through financial investment rather than labor, these contributions are not directly comparable with our model. We will return to this issue in Section 7, where scale effects will be tackled.

[17] Note that (15) encompasses also the case in which new knowledge is being produced by decentralized R&D-firms for a price $\eta(t) = \psi[A(t)]$ that includes a mark-up over the Tsur and Zemel (2007) first-best cost $\varphi[A(t)]$, as in Marchese et al. (2014).

Because from (16) $V(t) = \eta(t)$, equation (17) can be rewritten in the familiar form of a Hamilton-Jacobi-Bellman equation,

$$r(t) V(t) = \gamma L + \dot{V}(t), \tag{18}$$

in which the evolution through time of knowledge's value takes into account the assets' gains/ losses $\dot{V}(t) = \dot{\eta}(t)$ due to variations in the new knowledge's cost $\eta(t)$.

Using (17) in (2) we obtain the following time-dependent growth rate of consumption:

$$g(t) = \frac{\dot{C}(t)}{C(t)} = \frac{1}{\varepsilon_u[C(t)]}\left[\frac{\gamma L}{\eta(t)} + \frac{\dot{\eta}(t)}{\eta(t)} - \rho\right]. \tag{19}$$

As, by construction, this version of the model exhibits transition dynamics, we look for an Asymptotic Balanced Growth Path (ABGP) type of equilibrium. To this aim, we assume that asymptotically the intertemporal elasticity of substitution becomes constant, while the growth rate of the unit cost of new knowledge is required to vanish in the long-run; that is, we set $\lim_{t \to +\infty}[1/\varepsilon_u(C)] = 1/\sigma$, $\sigma > 0$, and $\lim_{t \to +\infty}[\dot{\eta}(t)/\eta(t)] = 0$. Note that this setting is sufficiently general to encompass any type of new knowledge production function, envisaging either increasing or decreasing $\eta(t)$ along transition dynamics, while asymptotically the unit cost of new knowledge must converge to some positive constant, $\lim_{t \to +\infty}\eta(t) = \eta^* > 0$, in order to satisfy the transversality condition for a solution of problem (1), as will be explained in Remark 1.

Proposition 4 *Under Assumptions 1, 3 and 4, assume that*
$\lim_{t \to +\infty}[1/\varepsilon_u(C)] = 1/\sigma$, $\sigma > 0$, *and* $\lim_{t \to +\infty}\eta(t) = \eta^* > 0$. *Then, if*

$$\gamma L > \rho\eta^* \qquad and \qquad (1-\sigma)\gamma L < \rho\eta^*, \tag{20}$$

then the economy admits a unique ABGP along which knowledge, output, and consumption all grow at the same asymptotic growth rate given by

$$g^* = \frac{\dot{C}}{C} = \frac{\dot{A}}{A} = \frac{\dot{Y}}{Y} = \frac{1}{\sigma}\left(\frac{\gamma L}{\eta^*} - \rho\right). \tag{21}$$

Conditions (20) and the growth rate (21) are the same as those in (13) and (12) respectively, only with the asymptotic value η^* in place of the constant unit cost η. Along the transition dynamics the consumption growth rate g in (19) may be either larger or smaller than its asymptotic value g^* in (21), depending on the (transition) intertemporal elasticity of substitution $1/\varepsilon_u(C)$, the (transition) knowledge cost η and the sign of $\dot{\eta}/\eta$, that is, on whether building new knowledge involves increasing or decreasing costs as time elapses.

Remark 1 *If we relax the assumption that* $\lim_{t \to +\infty}\eta(t) = \eta^* > 0$ *and allow either that* $\eta(t)$ *grows asymptotically,* $\lim_{t \to +\infty}[\dot{\eta}(t)/\eta(t)] > 0$ *with* $\eta(t) \to +\infty$ *as* $t \to +\infty$, *or that* $\eta(t)$ *keeps*

decreasing, $\lim_{t \to +\infty} [\dot{\eta}(t)/\eta(t)] < 0$ with $\eta(t) \to 0$ as $t \to +\infty$ (asymptotically vanishing cost of producing new ideas), an ABGP type of equilibrium cannot exist. To see this, suppose, on the contrary, that $\lim_{t \to +\infty} [\dot{\eta}(t)/\eta(t)] = g_\eta > 0$. Then $\gamma L/\eta(t) \to 0$ as $t \to +\infty$ and, from (17) and (19), $\lim_{t \to +\infty} r(t) = r^* \equiv g_\eta$ and $\lim_{t \to +\infty} g(t) = (g_\eta - \rho)/\sigma \equiv g^*$ respectively, so that, assuming $g_\eta > \rho$ to have $g^* > 0$ (positive asymptotic growth), one has $g_\eta + g^* = r^* + g^* > r^*$, which violates the transversality condition $\lim_{t \to +\infty} \eta(t) A(t) e^{-r^*t} = 0$.

The interpretation is that, according to (16), an increasing cost of knowledge production must be compensated by an increasing market value of knowledge, $V(t)$, a too heavy burden for the economy to sustain. On the other hand, $\lim_{t \to +\infty} [\dot{\eta}(t)/\eta(t)] = g_\eta < 0$ implies $\gamma L/\eta(t) \to +\infty$ as $t \to +\infty$, which would generate explosive growth and is thus incompatible with an ABGP of the sort defined by (21). In this case, vanishing costs of new knowledge production leads an immaterial economy like ours to outburst.

5   The Social Optimum

When the cost of knowledge varies through time depending on the evolution of the knowledge stock, externalities arise, but the households are not able to keep them into account in their decision process, as they can only observe the ensuing price changes, $\dot{\eta}(t)$, embedded in the interest rate according to (17). Clearly, under these circumstances the equilibrium described in Proposition 4 turns out to be not Pareto optimal, as we now show by solving the social planner problem associated to (1).

Now we assume that the unit cost of new knowledge production $\eta(t)$ has the form in (14) and, as in Proposition 4, $\lim_{t \to +\infty} \eta(t) = \lim_{t \to +\infty} \varphi[A(t)] = \eta^* > 0$ when there is knowledge growth, $\dot{A}(t)/A(t) > 0$. Following the same argument as in the proof of Proposition 3 in the Appendix, the social planner first maximizes net output, $Y(t) - X(t) = A(t)Lf[x(t)/A(t)] - x(t)L$, with respect to $x(t)$, obtaining the optimal net output $Y^S(t) - X^S(t) = \gamma LA(t)$. Then, she maximizes the representative household's lifetime discounted utility as in (1) subject to the resource constraint $\gamma LA(t) = C(t) + J(t)$, which, according to (14) and (15), can be written as

$$\dot{A}(t) = \frac{\gamma LA(t) - C(t)}{\varphi[A(t)]}. \tag{22}$$

Denoting by $\lambda(t)$ the costate variable associated to the unique dynamic constraint (22) and dropping the time argument for simplicity, the current-value Hamiltonian of the social planner problem is

$$H(A, C, \lambda) = u(C) + \lambda \frac{\gamma LA - C}{\varphi(A)}.$$

Necessary conditions are

$$u'(C) = \frac{\lambda}{\varphi(A)} \tag{23}$$

$$\dot{\lambda} = \rho\lambda - \lambda\frac{\gamma L\varphi(A) - (\gamma LA - C)\varphi'(A)}{[\varphi(A)]^2} \tag{24}$$

$$\lim_{t\to+\infty} \lambda(t) A(t) e^{-\rho t} = 0, \tag{25}$$

where (25) is the transversality condition. Differentiating with respect to time (23) one gets

$$\frac{\dot{\lambda}}{\lambda} = \frac{\varphi'(A)\dot{A}}{\varphi(A)} - \varepsilon_u(C)\frac{\dot{C}}{C}, \tag{26}$$

where $\varepsilon_u(C)$, as usual, denotes the inverse of the intertemporal elasticity of substitution. Coupling (26) with (24), using (22) and rearranging terms we obtain the following transitory consumption growth rate:

$$g^S(t) = \frac{\dot{C}(t)}{C(t)} = \frac{1}{\varepsilon_u[C(t)]}\left\{\frac{\gamma L}{\varphi[A(t)]} - \rho\right\}. \tag{27}$$

Also this version of the model exhibits transition dynamics; thus, we again look for an ABGP type of equilibrium which turns out to be the same as that characterized in Proposition 4.

Proposition 5 *Suppose that Assumptions 1, 3 and 4 with the specification in (14) hold. Moreover assume that $\lim_{t\to+\infty}\{1/\varepsilon_u[C(t)]\} = 1/\sigma$, $\sigma > 0$, and $\lim_{t\to+\infty}\varphi[A(t)] = \eta^* > 0$ whenever there is positive knowledge growth, $\dot{A}/A > 0$. Then, if conditions (20) hold, the social planner economy admits a unique ABGP along which knowledge, output, and consumption all grow at the same asymptotic growth rate characterized by the common constant growth rate as in (21) of Proposition 4 for the decentralized economy.*

Furthermore, along the transition dynamics the consumption growth rate in (27) can be either larger or smaller than that in (19), specifically, $g^S > g$ when $\varphi'[A(t)] < 0$, while $g^S < g$ when $\varphi'[A(t)] > 0$.

From Proposition 5 we conclude that, asymptotically, the equilibrium in the decentralized model of Section 4 converges to the Pareto optimal solution. However, along the transition dynamics the consumption growth rate under social planner supervision in (27) can be either larger or smaller than that in the decentralized market economy in (19), depending on the sign of $\varphi'[A(t)]$ which, in turn, determines the sign of the term $\dot{\eta}(t)/\eta(t) = \varphi'[A(t)]\dot{A}(t)/\varphi[A(t)]$ in the square bracket of the RHS in (19). Indeed, a social planner controls the whole evolution of the knowledge stock $A(t)$ so that, with a unit cost of new knowledge production as in (14), the externalities of investments in knowledge leading to changes in $\varphi[A(t)]$ through time are now taken into account.

Specifically, with costs that decrease in time, $\varphi'[A(t)] < 0$, the growth rate in the transitional period is larger under central control than in the decentralized model of Section 4: $g^S > g$. This is due to the presence of a positive externality, that is, when knowledge costs are decreasing in time, it becomes possible to produce subsequent inventions by subtracting less and less resources from other uses. This external effect, however, is not accounted for by private investors, while it is being considered by the central planner, who, accordingly, chooses a larger growth rate g^S in the transitional period.

Conversely, if $\varphi'[A(t)] > 0$, the growth rate in the transitional period is smaller under the social planner supervision than in the decentralized model: $g^S < g$.

As inefficiencies arise in the transitional period under decentralization, corrective policy interventions might be designed. That is, a subsidy to R&D investments along the transition when $\varphi'[A(t)] < 0$ (a tax when $\varphi'[A(t)] > 0$) could be introduced to align the behavior of decentralized agents to the path envisaged by the social planner. We do not pursue here the detailed specification of such tools.

6   A Parameterized Example

Assume households have an instantaneous CIES utility, $u(C) = \frac{C^{1-\sigma}-1}{1-\sigma}$, $\sigma > 0$, and F-firms production function has the Cobb-Douglas form, $Y = F(X, AL) = \theta X^\alpha (AL)^{1-\alpha} = \theta AL (x/A)^\alpha$, $\theta > 0$ and $0 < \alpha < 1$. Thus, $f(x/A) = \theta(x/A)^\alpha$ and parameters in (9) and (10) are given by $\delta = (\theta\alpha)^{1/(1-\alpha)}$ and $\gamma = \theta^{1/(1-\alpha)}\alpha^{\alpha/(1-\alpha)}(1-\alpha)$ respectively. Under Assumption 2 the unit cost of new knowledge production, η, is constant, and the interest rate in (11) is given by

$$r = \frac{\gamma L}{\eta} = \frac{\theta^{1/(1-\alpha)}\alpha^{\alpha/(1-\alpha)}(1-\alpha)L}{\eta},$$

while, according to Proposition 2, the economy growth rate common to all variables is

$$g = \frac{\dot{C}}{C} = \frac{\dot{A}}{A} = \frac{\dot{Y}}{Y} = \frac{1}{\sigma}\left[\frac{\theta^{1/(1-\alpha)}\alpha^{\alpha/(1-\alpha)}(1-\alpha)L}{\eta} - \rho\right],$$

whenever $\theta^{1/(1-\alpha)}\alpha^{\alpha/(1-\alpha)}(1-\alpha)L > \rho\eta$ and $(1-\sigma)\theta^{1/(1-\alpha)}\alpha^{\alpha/(1-\alpha)}(1-\alpha)L < \rho\eta$ hold.

To consider a time-dependent unit cost of new knowledge production, $\eta(t)$, as foreseen by Assumption 4 with the (14) specification, let

$$\eta(t) = \varphi[A(t)] = \frac{\beta}{A(t)} + \eta^*, \qquad \beta, \eta^* > 0.$$

As the function $\varphi(A)$ is decreasing in A, under positive knowledge growth, $\dot{A}/A > 0$, $\eta(t)$ is decreasing in time and $\lim_{t\to+\infty}\eta(t) = \lim_{A\to+\infty}\varphi(A) = \eta^* > 0$, so that the assumptions of

Proposition 4 hold. According to (15), the new knowledge production function is

$$\dot{A}(t) = \frac{J(t)}{\varphi[A(t)]} = \frac{A(t)}{\beta + \eta^* A(t)} J(t),$$

envisaging increasing knowledge spillovers as $(\partial/\partial A)[A/(\beta + \eta^* A)] = \beta/(\beta + \eta^* A)^2 > 0$, with decreasing returns as $(\partial^2/\partial A^2)[A/(\beta + \eta^* A)] = -2\beta\eta^*/(\beta + \eta^* A)^3 < 0$. From (17) we get the transition interest rate as

$$r(t) = \frac{\theta^{1/(1-\alpha)}\alpha^{\alpha/(1-\alpha)}(1-\alpha)LA(t)}{\beta + \eta^* A(t)} - \frac{\beta}{\beta + \eta^* A(t)}\frac{\dot{A}(t)}{A(t)},$$

and Proposition 4 predicts an ABGP characterized by the following growth rate common to all variables,

$$g^* = \frac{\dot{C}}{C} = \frac{\dot{A}}{A} = \frac{\dot{Y}}{Y} = \frac{1}{\sigma}\left[\frac{\theta^{1/(1-\alpha)}\alpha^{\alpha/(1-\alpha)}(1-\alpha)L}{\eta^*} - \rho\right] \tag{28}$$

whenever $\theta^{1/(1-\alpha)}\alpha^{\alpha/(1-\alpha)}(1-\alpha)L > \rho\eta^*$ and $(1-\sigma)\theta^{1/(1-\alpha)}\alpha^{\alpha/(1-\alpha)}(1-\alpha)L < \rho\eta^*$ hold.

If we let

$$\eta(t) = \varphi[A(t)] = \eta^*\left[1 - \frac{1}{A(t) + \beta}\right], \qquad \beta \geq 1 \text{ and } \eta^* > 0,$$

we have a function $\varphi(A)$ which is increasing in A so that, under positive knowledge growth, $\dot{A}/A > 0$, $\eta(t)$ is increasing in time and again $\lim_{t\to+\infty}\eta(t) = \lim_{A\to+\infty}\varphi(A) = \eta^* > 0$; moreover, with $\beta \geq 1$ and assuming $A(0) = A_0 > 0$, $\eta(0) = \eta^*[1 - 1/(A_0 + \beta)] > 0$. Hence, the assumptions of Proposition 4 still hold, but now the unit cost of new knowledge increases in time. According to (15), the new knowledge production function is

$$\dot{A}(t) = \frac{J(t)}{\varphi[A(t)]} = \frac{A(t) + \beta}{\eta^*[A(t) + \beta - 1]}J(t),$$

characterized by decreasing knowledge spillovers because

$$\frac{\partial}{\partial A}\left[\frac{A + \beta}{\eta^*(A + \beta - 1)}\right] = -\frac{1}{\eta^*(\beta + \eta^* A)^2} < 0;$$

however, now such spillovers occur at a decreasing negative rate as

$$\frac{\partial^2}{\partial A^2}\left[\frac{A + \beta}{\eta^*(A + \beta - 1)}\right] = \frac{2}{\eta^*(\beta + \eta^* A)^3} > 0.$$

From (17) we get the transition interest rate as

$$r(t) = \frac{\theta^{1/(1-\alpha)}\alpha^{\alpha/(1-\alpha)}(1-\alpha)L[A(t) + \beta]}{\eta^*[A(t) + \beta - 1]} + \frac{\dot{A}(t)}{[A(t) + \beta - 1][A(t) + \beta]},$$

and Proposition 4 again predicts the same ABGP constant growth rate common to all variables

given by (28) whenever $\theta^{1/(1-\alpha)}\alpha^{\alpha/(1-\alpha)}(1-\alpha)L > \rho\eta^*$ and $(1-\sigma)\theta^{1/(1-\alpha)}\alpha^{\alpha/(1-\alpha)}(1-\alpha)L < \rho\eta^*$ hold.

7 Scale Effects and Population Growth

It is evident from (12) and (21) that, as one expects, both versions of our simplified economy exhibit the *strong scale effect*—the growth rate of the economy increases in population size–as postulated by Jones (1995; 1999; 2005) for knowledge-based endogenous growth models. While in the original Romer (1990) model the strong scale effect is determined by full knowledge spillovers among researchers, in our setting this effect is present because labor plays the role of the "carrier" of the knowledge input factor in the final good production, so that the economy's growth rate turns out to be directly affected by the total royalty $\Gamma = \gamma L$, corresponding to the Lindahl pricing of knowledge.

Also in our scenario, however, a more realistic weak scale effect can hold if there are increasing technical difficulties in combining labor and knowledge. That is, besides the decreasing marginal returns to augmented labor that characterize the instantaneous production function in the final good production sector, the "labor augmentation" process may become more difficult through time. To take this feature into account, we add a "damping" term, $\Omega(t)$, to the second argument in the neoclassical production function of Assumption 3 by defining aggregate output at instant t as[18]

$$Y(t) = F\left[X(t), \frac{A(t)L(t)}{\Omega(t)}\right], \tag{29}$$

where $X(t)$ still denotes the aggregate intermediate good amount used in final production. Because our aim is to tackle scale effects, in the sequel it will be assumed that population—and thus workers—grows exogenously at a constant rate $n > 0$, *i.e.*, $\dot{L}(t) = nL(t)$. For now let us only assume that the term $\Omega(t)$ depends on time; a more precise characterization of $\Omega(t)$ will be specified later on. Note the role of the term $\Omega(t)$ in (29): it mirrors the same effect as in the new knowledge production functions of Segerstrom (1998) and Kuwahara (2013), except that here such term affects final rather than knowledge production.

7.1 Market equilibrium

As in Subsection 2.3 we rewrite (29) as

$$Y(t) = \frac{A(t)}{\Omega(t)}L(t)f\left[\frac{\Omega(t)x(t)}{A(t)}\right], \qquad \text{with } f(\cdot) = F(\cdot, 1), \tag{30}$$

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[18]As explained in Subsection 2.3, all producing firms in the economy use the same production function; hence, there is no reason to keep the index i and we can consider the representative F-firm.

where, as usual, $x(t) = X(t)/L(t)$ denotes per capita intermediate good. Dropping the time index for simplicity, FOC for the representative F-firm now become:

$$\frac{\partial Y}{\partial X} = f'\left(\frac{\Omega x}{A}\right) = 1 \tag{31}$$

$$\frac{\partial Y}{\partial A} = \frac{L}{\Omega}\left[f\left(\frac{\Omega x}{A}\right) - \left(\frac{\Omega x}{A}\right)f'\left(\frac{\Omega x}{A}\right)\right] = \frac{L}{\Omega}\gamma\left(\frac{\Omega x}{A}\right), \tag{32}$$

where in (32) the term $\gamma(\Omega x/A)$ still denotes the equilibrium royalty per augmented worker, which here depends on the ratio $(\Omega x/A)$ and is being further 'diminished' by the term Ω. Following the same arguments as in Section 3 it is readily seen that (31) implies the demand for the intermediate good x be still linear in A, but now it depends also on Ω, according to $x = \left[(f')^{-1}(1)/\Omega\right]A = (\delta/\Omega)A$. Replacing the constant $\delta = (f')^{-1}(1) = (\Omega x/A)$ in (32) we obtain the per capita willingness to pay for knowledge, which is still the constant given by (10): $\gamma(\Omega x/A) \equiv \gamma = f(\delta) - \delta f'(\delta) = f(\delta) - \delta$. Hence, according to (32), the total royalty at instant t is now given by

$$\Gamma(t) = \gamma\frac{L(t)}{\Omega(t)}, \tag{33}$$

which is no more a constant because both $L(t)$ and $\Omega(t)$ depend on time.

Under Assumption 4 the free-entry condition in (16) can be rewritten as:

$$V(t) = \int_t^{+\infty}\Gamma(v)e^{-\int_t^v r(s)\,ds}\,dv = \eta(t), \tag{34}$$

where now $\Gamma(t)$ is given by (33). Differentiating both sides with respect to time leads to

$$\dot{V}(t) = r(t)\int_t^{+\infty}\Gamma(v)e^{-\int_t^v r(s)\,ds}\,dv - \Gamma(t) = \dot{\eta}(t),$$

which, after substituting the integral with (34) and using (33), yields the interest rate

$$r(t) = \frac{\Gamma(t)}{\eta(t)} + \frac{\dot{\eta}(t)}{\eta(t)} = \frac{\gamma L(t)}{\eta(t)\Omega(t)} + \frac{\dot{\eta}(t)}{\eta(t)}. \tag{35}$$

Using (34), equation (35) can be rewritten in the familiar form of a Hamilton-Jacobi-Bellman equation:

$$r(t)V(t) = \frac{\gamma L(t)}{\Omega(t)} + \dot{V}(t). \tag{36}$$

Under the assumption of constant exogenous population growth it is convenient to restate the representative household's problem in per capita terms:

$$\max_{[c(t)]_{t=0}^{\infty}}\int_0^{+\infty}u[c(t)]e^{-(\rho-n)t}\,dt \tag{37}$$

$$\text{subject to } \dot{b}(t) = [r(t) - n]b(t) - c(t),$$

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where $u(\cdot)$ still satisfies Assumption 1, $c(t) = C(t)/L(t)$ and $b(t) = B(t)/L(t)$ denote per capita consumption and asset respectively, $r(t)$ is the market rate of returns on assets, and $n = \dot{L}(t)/L(t)$, with the additional constraint $0 \le c(t) \le r(t)b(t)$, for a given initial asset level $b(0) = b_0 > 0$. The associated Euler equation is now stated in terms of per capita consumption growth rate:

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\varepsilon_u[c(t)]}[r(t) - \rho], \qquad (38)$$

where again $1/\varepsilon_u(c) = -u'(c)/[u''(c)c]$ is the intertemporal elasticity of substitution. Using (35) it can be rewritten as

$$g_c(t) = \frac{\dot{c}(t)}{c(t)} = \frac{1}{\varepsilon_u[c(t)]}\left[\frac{\gamma L(t)}{\eta(t)\Omega(t)} + \frac{\dot{\eta}(t)}{\eta(t)} - \rho\right], \qquad (39)$$

from which it is apparent that this model exhibits transition dynamics.

From (39) it is clear that in order to achieve an ABGP in the long-run, besides the assumptions of Proposition 4, i.e., $\lim_{t\to+\infty}[1/\varepsilon_u(C)] = 1/\sigma > 0$, and $\lim_{t\to+\infty}\eta(t) = \eta^* > 0$ [which implies that $\dot{\eta}(t)/\eta(t) \to 0$ as $t \to +\infty$], now we need the additional restriction that $\lim_{t\to+\infty}\left[\dot{\Omega}(t)/\Omega(t)\right] = n$; that is, to have an ABGP the term $\Omega(t)$ asymptotically must grow at the same exogenous growth rate of population.

As, under the assumption that $\dot{\Omega}/\Omega = n$, the ratio L/Ω becomes constant in the long-run, it follows that also the ratio $c(t)/a(t)$ must eventually become constant (see the proof of the next Proposition 6 in the Appendix), that is, the asymptotic growth rate of per capita consumption must be equal to that of per capita knowledge: $\lim_{t\to+\infty}[\dot{c}(t)/c(t)] = \lim_{t\to+\infty}[\dot{a}(t)/a(t)] = \lim_{t\to+\infty}\left[\dot{A}(t)/A(t)\right] - n$. Therefore, in order to close the asymptotic equilibrium of our model we need to evaluate the long-run growth rate of knowledge, $\lim_{t\to+\infty}\left[\dot{A}(t)/A(t)\right]$; to this purpose a relationship between the damping term Ω and the stock of knowledge A is required.

A. 5 *The function $\Omega(t)$ depends on the stock of knowledge, i.e., $\Omega(t) = \Omega[A(t)]$, and has elasticity that becomes constant as $A \to +\infty$:*

$$\lim_{A\to+\infty}\frac{\Omega'(A)A}{\Omega(A)} = \varepsilon_\Omega^*, \qquad with\ 0 < \varepsilon_\Omega^* < 1.$$

Proposition 6 *Under the assumptions of Proposition 4 suppose that population grows according to a constant exogenous rate, $\dot{L}/L = n$, and Assumption 5 holds. Then, if*

$$\frac{1 - (1 - \varepsilon_\Omega^*)\sigma}{\varepsilon_\Omega^*}n < \rho, \qquad (40)$$

the economy admits a unique ABGP along which per capita consumption grows at the constant

asymptotic rate

$$g_c^* = \frac{\dot{c}}{c} = \left(\frac{1}{\varepsilon_\Omega^*} - 1\right) n, \tag{41}$$

while (aggregate) knowledge and output grow at the same constant asymptotic growth rate given by

$$g^* = \frac{\dot{A}}{A} = \frac{\dot{Y}}{Y} = \frac{n}{\varepsilon_\Omega^*}, \tag{42}$$

and the constant asymptotic interest rate is

$$r^* = \left(\frac{1}{\varepsilon_\Omega^*} - 1\right) \sigma n + \rho. \tag{43}$$

According to (41), Assumption 5 is necessary to have positive per capita consumption growth. In other words, the damping function $\Omega\left[A\left(t\right)\right]$ asymptotically must grow less than proportionally with respect to the stock of knowledge. This is consistent with the Jones (1995; 1999; 2005) and Kortum (1997) approach as, from (29), one can see that $\Omega\left(A\right)$ enters the denominator of a ratio having A in the numerator, thus contributing to slowing down the returns to knowledge for fixed labor L entering the production function of the final sector as time elapses.[19] As a consequence, (41) typically represents the growth rate of a "semi-endogenous" growth model as, if on one hand the economy exhibits sustained growth, on the other hand the per capita growth rate depends only on population growth and the technological parameter ε_Ω^* that determines how effectively knowledge is being employed in the final sector; specifically, the asymptotic growth rate cannot be affected by taxation or other policies. Interestingly, unlike the equilibria obtained in previous sections, here the per capita growth rate g_c^* does not depend either on preferences (parameters σ and ρ), the final good production function (parameter γ), or, most importantly, the new knowledge production function (parameter η^*).

Example 2 *Assume that $\Omega\left(A\right) = A^{1-\phi}$, with $0 < \phi < 1$. Then $\varepsilon_\Omega^* \equiv 1 - \phi$ for all A (and thus for all $t \geq 0$) and the second argument of the production function in (29) boils down to $A^\phi L$, resembling the familiar form of Jones (1995; 1999; 2005) new knowledge production function. Indeed, by Proposition 6, the long-run per capita consumption growth rate turns out to be*

$$g_c^* = \frac{\phi n}{1 - \phi}, \tag{44}$$

not much different than that found by Jones, which, in its simplified version, amounts to[20]

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[19]Note, however, that $\varepsilon_\Omega^* < 1$ rules out the case in which the ratio $A\left(t\right)/\Omega\left(t\right)$ is decreasing in time; this scenario corresponds to $\varepsilon_\Omega^* > 1$, which implies a negative per capita consumption asymptotic growth rate. In other words, to have positive per-capita consumption growth in the long-run a Segerstrom (1998) type of damping function $\Omega\left(A\right)$, reducing the productivity of labor as the stock of knowledge increases, cannot be applied in our setting.

[20]This value corresponds to that defined in equation (13.38) on p. 447 in Acemoglu (2009). In the original works of Jones (1995; 1999; 2005) also a parameter $0 < \lambda \leq 1$ indicating decreasing returns in researchers' effort appear in the numerator, yielding a per capita consumption growth rate given by $g_c^J = \lambda n/\left(1 - \phi\right)$.

$g_c^J = n/(1-\phi)$. *Note that the appearance of ϕ both in the numerator and in the denominator of (44) lets the consumption growth rate be more sensitive with respect to the technological parameter ϕ than it occurs in Jones's growth rate, as for values of ϕ close to 0 our economy exhibits a per capita consumption growth rate smaller than that in Jones, which vanishes as $\phi \to 0^+$. The intuition might be that while in Jones the technological parameter ϕ affects income growth by slowing only the productivity of researchers, in our case it affects the whole labor. It becomes clear at any rate that the insensitivity to standard policy tools, as subsidies to R&D, arises both here and in semi-endogenous growth models when inefficiencies involve labor.*

7.2 Centralized solution

In this section so far we considered a unit cost of knowledge production, $\eta(t)$, that changes through time along the transitory dynamics. In order to focus on the correction of the market failure generated by the damping function $\Omega(t)$ in (29), we resume Assumption 2 and assume that the unit cost of knowledge production is constant $\eta(t) \equiv \eta$; therefore, from Proposition 3 we know that no externalities are involved in the knowledge generation process and can examine exclusively how the function $\Omega(t)$ let the decentralized equilibrium characterized above depart from the optimal solution.

Under Assumptions 1–2, suppose that population grows exogenously at a constant rate $n > 0$, i.e., $\dot{L}(t) = nL(t)$, and total output is given by (30), with a damping function $\Omega(t) = \Omega[A(t)]$ satisfying assumption 5. As usual, the social planner first maximizes net output which, using (30) and $X(t) = x(t)L(t)$, amounts to

$$Y(t) - X(t) = \frac{A(t)}{\Omega(t)} L(t) f\left[\frac{\Omega(t) x(t)}{A(t)}\right] - x(t) L(t),$$

with respect to $x(t)$, obtaining the optimal net output $Y^S(t) - X^S(t) = \gamma A(t) L(t)/\Omega(t)$, with γ still defined by (10). Using (3) the dynamic resource constraint can be written as

$$\dot{A}(t) = \frac{1}{\eta}\left[\frac{\gamma L(t) A(t)}{\Omega[A(t)]} - C(t)\right],$$

where all variables, including consumption, are written in aggregate terms. Hence, the social planner problem associated to (37) is:

$$\max_{[c(t)]_{t=0}^{\infty}} \int_0^{+\infty} L(t) u[c(t)] e^{-\rho t} \, \mathrm{dt} \tag{45}$$

$$\text{subject to } \dot{A}(t) = \frac{L(t)}{\eta}\left[\frac{\gamma A(t)}{\Omega[A(t)]} - c(t)\right],$$

where $c(t) = C(t)/L(t)$ denotes per capita consumption, with the additional constraint $0 \le c(t) \le \gamma A(t)/\Omega[A(t)]$, for a given initial knowledge stock $A(0) = A_0 > 0$.

Denoting by $\lambda(t)$ the costate variable associated to the unique dynamic constraint and dropping the time argument for simplicity, the current-value Hamiltonian of the social planner problem is

$$H(A, C, \lambda) = Lu(c) + \frac{\lambda L}{\eta}\left[\frac{\gamma A}{\Omega(A)} - c\right].$$

Necessary conditions are

$$u'(c) = \frac{\lambda}{\eta} \tag{46}$$

$$\dot{\lambda} = \rho\lambda - \frac{\lambda\gamma L[\Omega(A) - A\Omega'(A)]}{\eta[\Omega(A)]^2} \tag{47}$$

$$\lim_{t\to+\infty}\lambda(t)A(t)e^{-\rho t} = 0, \tag{48}$$

where (48) is the transversality condition. Differentiating with respect to time (46) one gets the usual condition $\dot{\lambda}/\lambda = -\varepsilon_u(c)(\dot{c}/c)$, where $\varepsilon_u(c)$ denotes the inverse of the intertemporal elasticity of substitution, which, coupled with (47) and rearranging terms yields the following transitory consumption growth rate:

$$g_c^S = \frac{\dot{c}}{c} = \frac{1}{\varepsilon_u(c)}\left\{\frac{\gamma L}{\eta\Omega(A)}\left[1 - \frac{\Omega'(A)A}{\Omega(A)}\right] - \rho\right\}. \tag{49}$$

Also this version of the model exhibits transition dynamics; hence, again we look for an ABGP type of equilibrium.

Proposition 7 *Under the assumptions of Proposition 6, if condition (40) holds, the social planner economy admits a unique ABGP along which per capita consumption grows at the asymptotic constant growth rate given by (41), while knowledge and output grow at the asymptotic constant growth rate as in (42) of Proposition 6 for the decentralized economy.*

Furthermore, along the transition dynamics the consumption growth rate in (49) is larger, equal to, or smaller than that in (39), i.e., $g_c^S(t) > g_c(t)$, $g_c^S(t) = g_c(t)$, or $g_c^S(t) < g_c(t)$, provided that $\varepsilon_\Omega^*(t) = \Omega'[A(t)]A(t)/\Omega[A(t)]$ satisfies $\varepsilon_\Omega^*(t) < 0$, $\varepsilon_\Omega^*(t) = 0$, or $\varepsilon_\Omega^*(t) > 0$, respectively.

Proposition 7 states that, like in all previous versions of the model, asymptotically the equilibrium in the decentralized model characterized by Proposition 6 converges to the Pareto optimal solution. However, along the transition dynamics the consumption growth rate under social planner supervision in (49) may be larger or smaller than that in the decentralized market economy described by (39) depending on the transitory value of parameter $\varepsilon_\Omega^*(t)$. This is because a social planner takes into account the negative externalities associated to the function $\Omega[A(t)]$ in the final sector production process. Specifically, depending on whether the damping term $\Omega[A(t)]$ decreases or increases as knowledge A piles up through time—i.e., $\Omega'[A(t)] < 0$ or $\Omega'[A(t)] \geq 0$—the social planner is able to increase or decrease consumption growth

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accordingly, pushing it up when knowledge is expected to become more effectively employed by the final sector in the near future, that is, when the term $\Omega\left[A\left(t\right)\right]$ decreases, while it may be optimal even a negative consumption growth rate if $\varepsilon_\Omega^*\left(t\right)$ is sufficiently large, *i.e.*, when $\varepsilon_\Omega^*\left(t\right) > 1 - \rho\eta\Omega\left[A\left(t\right)\right]/\left[\gamma L\left(t\right)\right]$. Households, on the other hand, can only observe the effect of $\Omega\left[A\left(t\right)\right]$ through the interest rate, which, according to (35) when η is constant and thus $\dot{\eta}/\eta = 0$, is given by $r\left(t\right) = \left(\gamma/\eta\right)L\left(t\right)/\Omega\left[A\left(t\right)\right]$, and are not able to discount changes in $\Omega\left[A\left(t\right)\right]$ that will affect the future effectiveness of the knowledge factor in final production.

Note, however, that Assumption 5 implies that after some (perhaps large) instant $T > 0$, $\varepsilon_\Omega^*\left(t\right)$ must satisfy $\varepsilon_\Omega^*\left(t\right) > 0$ for all $t \geq T$. Therefore, after T and before reaching the ABGP—where, according to the first part Proposition 7, the asymptotic growth rate of consumption is the same both in the centralized and decentralized models—the social planner will choose a smaller consumption growth rate than in the decentralized equilibrium because, unlike households, she anticipates the increasing difficulty in knowledge usage by the final sector firms.

8   Conclusions

The model presented in this paper, in addition to being simple, is based on assumptions for the fundamentals of the economy that cover the broadest class of neoclassical environments. Specifically, unlike most of the existing literature (including some papers mentioned above) that requires peculiar functional forms like CIES (if not logarithmic) preferences for the representative consumer and Leontief or Cobb-Douglas technologies for the productive sector, our setting works for any increasing and concave utility function with constant asymptotic intertemporal elasticity of substitution and CRS production function exhibiting decreasing returns in each factor taken alone. Moreover, also the creation of new knowledge in the R&D sector occurs according to a sufficiently general family of technologies that include those already considered in the literature; the only restriction is that these technologies asymptotically must envisage a constant unit cost of production. In this setting we take into account the increasingly immaterial characteristics assumed by technical progress and the implications this has on the ways in which it is transferred to final production. On this basis we thus assume that both a direct use of ideas in final goods' production and the revelation of firm's willingness to pay for accessing knowledge arise. A further feature that characterizes our model is the reconsideration of the role of labor. In our scenario cognitive tasks—traditionally performed by medium or even high skilled workers—are embedded into routines and software, and are thus automated. The truly creative activities are done by research firms under the protection of IPR. As for the tasks traditionally performed by low-skilled workers, they too are considered as based on models, whose routines, however, are not (yet) been codified. Firms demand these routines "embedded" in raw labor. Thus, both workers and patent holders supply the same type of input—*i.e.*, ideas, models of behavior, knowledge—while workers provide them jointly with raw labor. As knowledge growth occurs through investments in patented innovations, an important implication of

the model is a tendency toward the compression of incomes paid to workers—which can reduce to the sole compensation of the basic competencies—while the remaining income goes to patent holders.

We show that when one takes into account the aforementioned trends of recent technical progress, competition becomes viable. Hence the economy can reach the first best if knowledge, while being nonrival, is homogeneous and fully excludable when used in final goods production. Even if partial excludability occurs, second best results would be confined to the transitional period, while first best is reached all the same in the long run. Even though non competitive markets in practice are often observed, stressing that competition is logically viable has important policy implications, since it means that, *e.g.*, regulatory and judicial interventions in the field of patents and IPR can foster competition without fearing that a collapse of research activities arises.

While the potential unwanted effects of technical progress have been often identified with the possible growth in unemployment due to the substitution of capital for labor, these dire effects did not materialize in the last decades in advanced countries, where unemployment—bar for the financial crisis years—has not been the more worrying problem. One of the prominent social concerns has actually been instead the shrinking of labor income share. The contribution of this paper is a possible rationale for this stylized fact, which does not fit well into standard growth models where the stability of factors' income shares is a tenet. As long as development disempowers the role of human capital, it raises also potential severe distributive problems. Our simple setting, however, is not suitable for studying them, and thus they are left for future research.

Appendix

Proof of Proposition 2. Clearly, the first condition in (13) implies that, according to (12), $g = \dot{C}/C > 0$. Differentiating with respect to time $\ln(Y)$, with Y as in (4), and recalling that, from (9), $x/A \equiv \delta$ is constant, it is immediately seen that $\dot{Y}/Y = \dot{A}/A$. Recall that, from (8), $B = VA = \eta A$; hence, as under Assumption 2 both η and, by (11), the interest rate, $r = \gamma L/\eta$, are constant the instantaneous budget constraint in (1) can be rewritten as

$$\frac{\dot{A}}{A} = \frac{1}{\eta}\left(\gamma L - \frac{C}{A}\right), \tag{50}$$

which implies that, in order to $\dot{A}/A$ be constant along the BGP, the ratio C/A on the RHS must be constant as well, which is possible if and only if $\dot{A}/A = \dot{C}/C = g$. Next, note that the second condition in (13) implies that $r > g = \dot{A}/A$, so that the transversality condition for problem (1), $\lim_{t\to+\infty} B(t)e^{-rt} = \lim_{t\to+\infty} \eta A(t)e^{-rt} = 0$, holds. Finally, for each $A(t)$ the amount of the intermediate good is given by (9) as $x(t) = \delta A(t)$; therefore, in $t = 0$, $x(0) = \delta A_0$ and the economy is immediately put on the BGP. ∎

Proof of Proposition 3. The resource constraint of the economy at instant t is

$$C(t) + J(t) = Y(t) - X(t), \tag{51}$$

where on the RHS we consider total output net of the intermediate goods, $X(t) = x(t)L$. Dropping time dependency for simplicity, in order to obtain a dynamic constraint in the only variables A (state) and C (control) a social planner first considers maximization of the net output $Y - X = [Af(x/A) - x]L$ with respect to x for a given stock A at instant t: the solution is $x^S = (f')^{-1}(1)A = \delta A$, where the superscript "S" denotes the level of per capita intermediate good chosen by the social planner, which happens to be the same as in (9). Hence, net output turns out to be $Y^S(t) - X^S(t) = L[f(\delta) - \delta]A(t) = \gamma LA(t)$, where in the last equality we used (10). Under Assumption 2 $J = \eta \dot{A}$, and (51) can be rewritten as

$$\dot{A}(t) = \frac{\gamma(t)LA(t) - C(t)}{\eta},$$

which, as $B(t) = \eta A(t)$ with η constant, turns out to be the same as the household's budget constraint in (1). Hence, the social planner problem is the same as (1) and has the same equilibrium of Proposition 2 as solution. ∎

Proof of Proposition 4. The arguments are the same as in the previous proof of Proposition 2 and thus we omit them. The only difference is that now from (16) $B(t) = V(t)A(t) = \eta(t)A(t)$ holds, so that $\dot{B}(t) = \dot{\eta}(t)A(t) + \eta(t)\dot{A}(t)$; however, it is immediately seen that the instantaneous budget constraint in (1) at each instant t remains the same as in (50), as, using (17) and rearranging terms,

$$\frac{\dot{A}(t)}{A(t)} = \frac{\gamma L}{\eta(t)} + \frac{\dot{\eta}(t)}{\eta(t)} - \frac{\dot{\eta}(t)}{\eta(t)} - \frac{C(t)}{\eta(t)A(t)} = \frac{1}{\eta(t)}\left[\gamma L - \frac{C(t)}{A(t)}\right].$$

When $\lim_{t\to+\infty}\eta(t) = \eta^* > 0$, $\dot{\eta}(t) \to 0$ as $t \to +\infty$ and, according to (17), $\lim_{t\to+\infty}r(t) = r^* \equiv \gamma L/\eta^*$; thus, the second condition in (20) implies that $r^* > g^* = \dot{A}/A$, with g^* defined in (21), so that the transversality condition for problem (1), $\lim_{t\to+\infty}\eta(t)A(t)e^{-r(t)t} = \lim_{t\to+\infty}\eta^*A(t)e^{-r^*t} = 0$, holds. ∎

Proof of Proposition 5. When $\dot{A}/A > 0$, $\lim_{t\to+\infty}\varphi[A(t)] = \eta^* > 0$ implies that $\lim_{t\to+\infty}\varphi'[A(t)] = 0$, so that it is immediately seen that, as $\lim_{t\to+\infty}[1/\varepsilon_u(C)] = 1/\sigma$, the consumption growth rate in (27) asymptotically converges to that defined in (21). As asymptotically the dynamic constraint (22) becomes equal to (50), the same argument as in the proofs of Proposition 2 applies to establish that $\dot{Y}/Y = \dot{A}/A = \dot{C}/C = g^*$ while the first condition in (20) implies that, according to (21), $g^* > 0$. The second condition in (20) is equivalent to $(1-\sigma)g^* < \rho$, which, according to (26) and, under (14), $\lim_{t\to+\infty}[\dot{\eta}(t)/\eta(t)] =$

$\lim_{t\to+\infty}\left\{\varphi'\left[A\left(t\right)\right]\dot{A}\left(t\right)/\varphi\left[A\left(t\right)\right]\right\}=0$, implies

$$\lim_{t\to+\infty}\left(\frac{\dot{\lambda}}{\lambda}+\frac{\dot{A}}{A}\right)=\lim_{t\to+\infty}\left(\frac{\dot{\eta}}{\eta}-\sigma\frac{\dot{C}}{C}+\frac{\dot{A}}{A}\right)=\left(1-\sigma\right)g^*<\rho,$$

thus ensuring that the transversality condition (25) holds. ∎

Proof of Proposition 6. Under the assumptions $\lim_{t\to+\infty}\left[1/\varepsilon_u\left(c\right)\right]=1/\sigma>0$ and $\lim_{t\to+\infty}\eta\left(t\right)=\eta^*>0$ [implying $\dot{\eta}\left(t\right)/\eta\left(t\right)\to0$ as $t\to+\infty$], from (39) it is clear that, in order to achieve an ABGP in the long-run, additionally

$$\lim_{t\to+\infty}\frac{\dot{\Omega}\left(t\right)}{\Omega\left(t\right)}=\frac{\dot{L}\left(t\right)}{L\left(t\right)}=n \tag{52}$$

must hold. In view of (30) and recalling that the argument of the function $f\left(\cdot\right)$ is constant, $\left(\Omega x/A\right)=\delta$, it is immediately seen that, under (52), on the ABGP aggregate output grows at the same growth rate of knowledge:

$$\frac{\dot{Y}\left(t\right)}{Y\left(t\right)}=\frac{\partial}{\partial t}\left[\ln A\left(t\right)-\ln\Omega\left(t\right)+\ln L\left(t\right)+\ln f\left(\delta\right)\right]=\frac{\dot{A}\left(t\right)}{A\left(t\right)}-\frac{\dot{\Omega}\left(t\right)}{\Omega\left(t\right)}+\frac{\dot{L}\left(t\right)}{L\left(t\right)}$$
$$=\frac{\dot{A}\left(t\right)}{A\left(t\right)}-n+n=\frac{\dot{A}\left(t\right)}{A\left(t\right)}. \tag{53}$$

From (34) we obtain the per capita asset as a function of per capita knowledge: $b\left(t\right)=B\left(t\right)/L\left(t\right)=V\left(t\right)A\left(t\right)/L\left(t\right)=\eta\left(t\right)a\left(t\right)$, where $a\left(t\right)=A\left(t\right)/L\left(t\right)$. Thus, the per capita household's budget constraint in (37) can be rewritten as

$$\frac{\dot{a}\left(t\right)}{a\left(t\right)}=r\left(t\right)-n-\frac{\dot{\eta}\left(t\right)}{\eta\left(t\right)}-\frac{c\left(t\right)}{\eta\left(t\right)a\left(t\right)}=\frac{\gamma L\left(t\right)}{\eta\left(t\right)\Omega\left(t\right)}+\frac{\dot{\eta}\left(t\right)}{\eta\left(t\right)}-n-\frac{\dot{\eta}\left(t\right)}{\eta\left(t\right)}-\frac{c\left(t\right)}{\eta\left(t\right)a\left(t\right)}$$
$$=\frac{\gamma L\left(t\right)}{\eta\left(t\right)\Omega\left(t\right)}-n-\frac{c\left(t\right)}{\eta\left(t\right)a\left(t\right)}, \tag{54}$$

where in the second equality we used (35). Because, by Assumption 5,

$$\lim_{t\to+\infty}\frac{\dot{\Omega}\left(t\right)}{\Omega\left(t\right)}=\lim_{t\to+\infty}\frac{\Omega'\left[A\left(t\right)\right]\dot{A}\left(t\right)}{\Omega\left[A\left(t\right)\right]}=\lim_{t\to+\infty}\left\{\frac{\Omega'\left[A\left(t\right)\right]A\left(t\right)}{\Omega\left[A\left(t\right)\right]}\cdot\frac{\dot{A}\left(t\right)}{A\left(t\right)}\right\}$$
$$=\lim_{t\to+\infty}\frac{\Omega'\left[A\left(t\right)\right]A\left(t\right)}{\Omega\left[A\left(t\right)\right]}\lim_{t\to+\infty}\frac{\dot{A}\left(t\right)}{A\left(t\right)}=\varepsilon_\Omega^* g^*,$$

joining it with (52) from (53) one immediately gets (42). It follows from (54) that $g_c^*=\lim_{t\to+\infty}\left[\dot{c}\left(t\right)/c\left(t\right)\right]=g^*-n=\left(1/\varepsilon_\Omega^*-1\right)n$, which is (41). The interest rate in (43) is immediately obtained using (41) in the (asymptotic) Euler equation (38). Finally, using (43), condition (40) implies that $r^*>g^*=\dot{A}/A$, so that the transversality condition for problem

(37), $\lim_{t\to+\infty} \eta(t) A(t) e^{-r(t)t} = \lim_{t\to+\infty} \eta^* A(t) e^{-r^* t} = 0$, holds. ∎

Proof of Proposition 7. Under the same assumptions as in Proposition 6, specifically under Assumption 5, from (49) it follows that, in order to achieve an ABGP in the long-run, $\Omega(t)$ and $L(t)$ must grow at the same constant rate, n, so that g^* as in (42) is immediately obtained. Rewriting the dynamic resource constrain of (45) as

$$\frac{\dot{A}(t)}{A(t)} = \frac{1}{\eta}\left[\frac{\gamma L(t)}{\Omega[A(t)]} - \frac{L(t) c(t)}{A(t)}\right],$$

it is immediately seen that $\dot{A}/A$ is constant along the ABGP only if the ratio $L(t) c(t)/A(t)$ on the RHS is constant as well, which is possible only if $\dot{A}/A = g^* = \dot{L}/L + \dot{c}/c = n + g_c^*$, which, using (42), yields (41). Using (47) and (42), it holds

$$
\lim_{t\to+\infty}\left(\frac{\dot{\lambda}}{\lambda} + \frac{\dot{A}}{A}\right) = \lim_{t\to+\infty}\left\{\rho - \frac{\gamma L}{\eta\Omega(A)}\left[1 - \frac{\Omega'(A) A}{\Omega(A)}\right]\right\} + \lim_{t\to+\infty}\frac{\dot{A}}{A}
$$
$$
= \lim_{t\to+\infty}\left[\rho - \frac{\gamma L}{\eta\Omega(A)}(1 - \varepsilon_\Omega^*)\right] + \frac{n}{\varepsilon_\Omega^*}. \tag{55}
$$

On the other hand, from (49), asymptotically it holds

$$
\lim_{t\to+\infty}\frac{1}{\varepsilon_u(c)}\left\{\frac{\gamma L}{\eta\Omega(A)}\left[1 - \frac{\Omega'(A) A}{\Omega(A)}\right] - \rho\right\} = \frac{1}{\sigma}\lim_{t\to+\infty}\left[\frac{\gamma L}{\eta\Omega(A)}(1 - \varepsilon_\Omega^*) - \rho\right] = g_c^*,
$$

from which, using (41), one gets

$$
\sigma g_c^* = \lim_{t\to+\infty}\left[\frac{\gamma L}{\eta\Omega(A)}(1 - \varepsilon_\Omega^*) - \rho\right] = \sigma\left(\frac{1}{\varepsilon_\Omega^*} - 1\right)n
$$

which, substituting into (55), yields

$$
\lim_{t\to+\infty}\left(\frac{\dot{\lambda}}{\lambda} + \frac{\dot{A}}{A}\right) = -\sigma\left(\frac{1}{\varepsilon_\Omega^*} - 1\right)n + \frac{n}{\varepsilon_\Omega^*} = \frac{1 - (1 - \varepsilon_\Omega^*)\sigma}{\varepsilon_\Omega^*}n,
$$

so that the transversality condition (48) holds whenever condition (40) is satisfied.

Finally, recall that under Assumption 2 $\dot{\eta}/\eta = 0$, so that (39) becomes

$$
g_c(t) = \frac{1}{\varepsilon_u[c(t)]}\left\{\left(\frac{\gamma}{\eta}\right)\frac{L(t)}{\Omega[A(t)]} - \rho\right\},
$$

which is clearly smaller, equal to, or larger than $g_c^S(t)$ in (49) whenever $\varepsilon_\Omega^*(t) < 0$, $\varepsilon_\Omega^*(t) = 0$, or $\varepsilon_\Omega^*(t) > 0$, respectively. ∎

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