Search frictions and (in)efficient vocational training over the life-cycle

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Abstract

This paper examines life cycle vocational training investments in the context of a model with search frictions. We emphasize that related externalities are age-dependent, and this can require an hump-shaped subsidy rate of training costs to restore social efficiency. These results are illustrated using a calibration on the french economy.

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1 Introduction

Since Becker [1964] it is well known that, in a context of competitive markets, human capital investments are in general efficient. On the opposite, search frictions on the labor market give rise to some inefficiency issues, so that there is a room for an optimal policy to promote vocational training investments. More particularly, Acemoglu [1997] and Acemoglu and Pischke [1999a] highlighted the impact of a poaching externality: as general human capital investments can benefit, with some probability, to some future employers, the current firm's private return of training investment is lower than its social return. Therefore, some subsidies can be required to reach a higher level of training expenditures. Recently, Belan and Chéron [2014] also argued that due to a higher job finding rate of workers with a higher general human capital, vocational training of workers accounts for an additional unemployment externality: the social return of training indeed embodies the fact that unemployed worker with higher human capital will switch faster from home production to market production.

This paper develops a life cycle approach to vocational training investments in the context of search frictions. Our main goal is actually to examine to what extent the design of training subsidies is life cycle dependent. This seems all the more important that life cycle issues for a labor market with search frictions have been pointed out by recent works. Chéron et al. [2011, 2013] for instance emphasized distance-to-retirement effects on workers' flows and showed that there exists an age-specific externality related to the job creation process. Menzio et al. [2015] provided a similar life cycle approach with human capital accumulation to predict US labor market flows and wage growth, but with an exogenous accumulation process. Lastly, Messe and Rouland [2014] built on Chéron et al. [2011] to propose a model with search frictions and endogenous human capital accumulation, in order to examine how training investments are related to age-dependent employment protection. But, as this paper deals with specific human capital accumulation, it does not raise any age-dependent externality. In turn, in this paper, we examine how search frictions and externalities related to training investments in general human capital can interact each other over the life cycle.

On the one hand, this allows to emphasize that, with respect to what would be optimal to do, firms typically reduce too far from retirement the entry of workers into training programs. Otherwise stated, some older workers that firms do not to train would have to be trained when poaching and unemployment externalities are internalized. But on the other hand, it comes that both private return and social return of training investments collapse to zero when retirement gets closer. Overall, there exists two opposite forces as worker is aging, so that this can require an hump-shaped age-dynamics of training subsidies to restore social efficiency. Those results are discussed both by having some theoretical insights and using an illustrative calibration on the french economy.

The remaining of the paper is then organized as follows. Section 2 presents the benchmark model and is doing some comparative statics. We lay out the economic environment and characterize both equilibrium and optimal age-dynamics of training policy to identify related externalities over the life cycle. Section 3 implements an empirical investigation, focusing on training age-dynamics and deriving the optimal age-profile of the subsidy rate of training. A final section concludes.

2 Equilibrium and efficient vocational training in a life cycle model with search frictions

Overall, the model features search frictions and general human capital depreciation during unemployment spell, which depends on a "turbulence" parameter in line with what have been pointed out by Ljungqvist and Sargent [1998]. Human capital accumulation relates to firms' endogenous decision to train workers at the time of hiring, as in the paper by Belan and Chéron [2014]. We actually extend the latter to account for a finite horizon, and this leads to an age-dependent selection of workers by firms into vocational training programs, that is due to a distance-to-retirement effect. Yet, we consider exogenous job finding probabilities that are not age-related, whereas we obviously know that it should be. Our point here is indeed to focus on the age-dynamics of training policies taking as given search frictions.[1]

2.1 Economic environment

Workers are characterized by their ability level, denoted by a, distributed over the interval $[\underline{a}, \bar{a}]$ according to p.d.f. $f(a)$, and by their age, denoted by t. The model is in discrete

[1] In a companion paper we address this issue of the joint age-dynamics of endogenous training and recruitment decisions. This allows to show that the sensitivity of the job finding rate to the distance-to-retirement is enhanced for workers that face lower human capital. Our main results are then reinforced.

time and at each period the older worker generation retiring from the labor market is replaced by a younger generation of the same size (normalized to unity) so that the population on the labor market is constant. We assume that each worker of the new generation enters the labor market at age $t = 0$ as unemployed and retires at a deterministic age T.

We assume that workers enter the labor market with up-to-date knowledge, so they get the highest level of general human capital hence related productivity $(1 + \Delta)a$. But they can face skill obsolescence (human capital depreciation) over the life cycle: this occurs during unemployment spells with a per period probability π; if so does, productivity of those workers is only given by a. Then, at the time of hiring, firms can choose to train workers whose human capital have been depreciated, in order to restore productivity $(1 + \Delta)a$ instead of a. This leads firms to bear an instantaneous training cost γ_f. Obviously, this intertemporal decision will depend on workers' ability, so that the training policy will consist in determining an ability threshold that is age-dependent, denoted $\tilde{a}_t$.

Therefore, workers are heterogenous according to three dimensions: (i) ability a, (ii) age t, and (iii) status wrt. skill obsolescence. This implies in particular that we need to distinguish three types of agents at the time of hiring:

- Type-0, with obsolete knowledge and unable for training ($a < \tilde{a}_t$), with expected instantaneous productivity a;
- Type-1, able for training ($a \geq \tilde{a}_t$) but with obsolete knowledge, with expected instantaneous productivity $(1 + \Delta)a$ once the cost γ_f will have been paid;
- Type-2, with up-to-date skills with expected instantaneous productivity $(1 + \Delta)a$ without any additional cost.

Furthermore, we consider a partial equilibrium framework where the frictional labor market is featured by exogenous job finding probabilities, constant across ages. The probability for an unemployed worker of age t to be employed at age $t + 1$ is assumed to be given by:

- p_0 for individuals with obsolete skills, unable for training if they are hired at the next period ($a < \tilde{a}_{t+1}$)
- p for individuals with up-to-date skills or individuals with obsolete knowledge but able for training if they are hired at the next period ($a \geq \tilde{a}_{t+1}$)

with $p_0 \leq p$.[2] This means that worker's status with respect to skill obsolescence is observed by the employer only after the match is formed, whereas the ability a is perfectly known *ex-ante*. Related labor market flows are provided in Appendix A.

2.2 Value functions and Nash bargaining

Let $w_{j,t}(a)$ be the wage, β the discount factor, δ the job destruction probability and b the domestic production. The expected values of income streams, denoted by $E_{j,t}(a)$ for a worker and $U_{j,t}(a)$ for an unemployed of type j and age t, are defined $\forall\ t < T-1$ by :[3]

Type 0:

$$E_{0,t}(a) = w_{0,t}(a) + \beta \left[(1-\delta)\, E_{0,t+1}(a) + \delta U_{0,t+1}(a)\right] \quad ,\forall\ a < \tilde{a}_t$$

$$U_{0,t}(a) = b + \beta \left[p_0 E_{0,t+1}(a) + [1-p_0]\, U_{0,t+1}(a)\right] \quad ,\forall\ a < \tilde{a}_t$$

Type 1:

$$E_{1,t}(a) = w_{1,t}(a) + \beta \left[(1-\delta)\, E_{1,t+1}(a) + \delta U_{2,t+1}(a)\right] \quad ,\forall\ a \geq \tilde{a}_t$$

$$U_{1,t}(a) = b + \beta \begin{cases} pE_{1,t+1}(a) + [1-p]\, U_{1,t+1}(a) & ,\ \forall\ a \geq \tilde{a}_{t+1} \\ p_0 E_{0,t+1}(a) + [1-p_0]\, U_{0,t+1}(a) & ,\ \forall\ a \in [\tilde{a}_t; \tilde{a}_{t+1}[\end{cases}$$

Type 2:

$$E_{2,t}(a) = w_{2,t}(a) + \beta \left[(1-\delta)\, E_{2,t+1}(a) + \delta U_{2,t+1}(a)\right] \quad ,\forall\ a$$

$$U_{2,t}(a) = b + \beta \left[pE_{2,t+1}(a) + [1-p]\,(1-\pi) U_{2,t+1}(a) + [1-p]\,\pi \begin{cases} U_{1,t+1}(a) & ,\ \forall\ a \geq \tilde{a}_{t+1} \\ \\ U_{0,t+1}(a) & ,\ \forall\ a < \tilde{a}_{t+1} \end{cases} \right]$$

[2] As already emphasized by Belan and Chéron [2014] in a conventional infinitely-lived agent model, such a ranking for probabilities typically emerges as an equilibrium outcome with endogenous recruitment decisions. In addition, this leads those probabilities to depend on a ability a, but does not have any qualitative incidence on the efficiency and policy analysis. In a finite horizon context, we should also observe that both probabilities decrease with age. Nevertheless, this paper aims at focusing on age-dependent training externality and we choose to take as given labor market transition probabilities (search frictions).

[3] In T-1, we assume that all individuals expect to earn at the next period a universal retirement pension, noted R.

Value functions for unemployed of type 1 and 2 deserve further discussion. Indeed, it can be the case that the ability of workers is high enough at age t to be trained, but no longer at age $t+1$, so that they switch from type-1 to type-0 from t to $t+1$ (see the expression $U_{1,t}(a)$). Similarly, type-2 workers that remain unemployed and face human capital depreciation can directly switch to the 0-type, if $a < \tilde{a}_{t+1}$. We should also notice that type-1 employed workers switch to type-2, only once they experience an unemployment spell. Obviously, this raises a conventional hold-up issue: once a type-1 worker has been trained, he gets some incentives to ask for wage $w_{2,t}(a)$ instead of $w_{1,t}(a)$ which is lower in equilibrium. This adds another source of inefficiency (see also Belan and Chéron [2014] for a discussion). We choose to skip this issue to focus on age-dependent externalities and their implications for public training policy.

Turning to the expected values of filled jobs by a worker of age t and ability a, we have the following value functions, $\forall t < T-1$

$$
\begin{aligned}
J_{0,t}(a) &= a - w_{0,t}(a) + \beta\,(1-\delta)\,J_{0,t+1}(a) \\
J_{1,t}(a) &= (1+\Delta)a - w_{1,t}(a) + \beta\,(1-\delta)\,J_{1,t+1}(a) \\
J_{2,t}(a) &= (1+\Delta)a - w_{2,t}(a) + \beta\,(1-\delta)\,J_{2,t+1}(a)
\end{aligned}
$$

and some terminal conditions $J_{0,T-1}(a) = a - w_{0,T-1}(a)$, $J_{1,T-1}(a) = (1+\Delta)a - w_{1,T-1}$ and $J_{2,T-1}(a) = (1+\Delta)a - w_{2,T-1}$.

Then, we consider standard Nash bargaining of wages, and let α be the bargaining power of workers. Therefore, wages are derived from the following sharing rules:[4]

$$
\begin{aligned}
(1-\alpha)[E_{0,t}(a) - U_{0,t}(a)] &= \alpha J_{0,t}(a) \\
(1-\alpha)[E_{1,t}(a) - U_{1,t}(a)] &= \alpha[J_{1,t}(a) - \gamma_f] \\
(1-\alpha)[E_{2,t}(a) - U_{2,t}(a)] &= \alpha J_{2,t}(a)
\end{aligned}
$$

This implies the following wage equations, $\forall t < T-1$:

$$
\begin{aligned}
w_{0,t}(a) &= \alpha a + (1-\alpha)b + \alpha\beta p_{0,t}(a) J_{0,t+1}(a) \\
w_{1,t}(a) &= \alpha\left[(1+\Delta)a - \gamma_f(1-\beta(1-\delta))\right] + (1-\alpha)b + \alpha\beta p_{0,t}(a) J_{0,t+1}(a) - \beta(1-\alpha)\left[U_{2,t+1}(a) - U_{0,t+1}(a)\right] \\
w_{2,t}(a) &= \alpha(1+\Delta)a + (1-\alpha)b + \alpha\beta p_t(a) J_{2,t+1}(a) - \beta\pi(1-\alpha)\left[1 - p_t(a)\right]\left[U_{2,t+1}(a) - U_{0,t+1}(a)\right]
\end{aligned}
$$

[4] See Appendix B for more details.

2.3 Equilibrium training policy

The firm's training policy consists in determining the age-specific ability threshold $\tilde{a}_t$. Above this threshold, at the time of hiring the employer trains any worker that faced human capital depreciation during the unemployment spell. Hence, $\tilde{a}_t$ satisfies the following condition:

$$J_{1,t}(\tilde{a}_t) - \gamma_f = J_{0,t}(\tilde{a}_t)$$

This problem can be solved recursively starting from terminal condition at $t = T - 1$ (see Appendix C), and it comes that:

$$\Delta\tilde{a}_t = \frac{\gamma_f - \left\{\sum_{i=0}^{T-3-t} \beta\delta[\beta(1-\delta)]^i \left[U_{2,t+1+i}(\tilde{a}_t) - U_{0,t+1+i}(\tilde{a}_t)\right]\right\}}{\sum_{i=0}^{T-1-t}[\beta(1-\delta)]^i}, \ \forall\, t \leq T-3 \quad (1)$$

which in particular leads to:

$$\begin{aligned}
\Delta\tilde{a}_{T-1} &= \gamma_f \\
\Delta\tilde{a}_{T-2} &= \frac{\gamma_f}{\sum_{i=0}^{1}[\beta(1-\delta)]^i} \\
\Delta\tilde{a}_{T-3} &= \frac{\gamma_f - (\tilde{a}_{T-3} - b)\alpha\beta^2\delta[p_{T-2} - p_{0,T-2}]}{\sum_{i=0}^{2}[\beta(1-\delta)]^i + \alpha\beta p_{T-2}}
\end{aligned}$$

- For $T-1$, the condition is static and shows that the instantaneous productivity gain must be at least equal to training expenditures.
- For $T-2$, there exists a capitalization effect that depends both on the discount factor and the probability of job destruction, since productivity gain can last two periods with probability $1-\delta$.
- Then, $\forall t \leq T-3$, the larger the unemployment gap $[U_{2,t+1} - U_{0,t+1}]$, the lower the wage $w_{1,t}(a)$ and the ability threshold $\tilde{a}_t$ (see equation (1)). Indeed, while an unemployed worker with up-to-date skills (type-2) can benefit from productivity $(1+\Delta)a$ without incurring any training cost, type-1 workers that faced human capital depreciation agree with wage cuts to become of type-2 in the future. Such a wage cut is as much important as $[U_{2,t+1} - U_{0,t+1}]$ is high, because the latter gives the relative value of training for unemployed workers who undergo skill obsolescence.

2.4 Efficient training policy

The efficient training policy is now derived by considering social values of the workers, according to age, ability and types (0, 1 or 2). Those values are defined as follows. The social value of an employed worker with up-to-date skills (type 2) is:

$$\tilde{Y}_t(a) = (1+\Delta)a + \beta\left[(1-\delta)\tilde{Y}_{t+1}(a) + \delta Y^{u2}_{t+1}(a)\right] \quad ,\forall\ a$$

where the social value of an unemployed with up-to-date skills (type 2) satisfies:

$$Y^{u2}_t(a) \quad = \quad b + \beta\left[p\tilde{Y}_{t+1}(a) + [1-p](1-\pi)Y^{u2}_{t+1}(a) + [1-p]\pi\left\{\begin{array}{ll} Y^{u1}_{t+1}(a) & ,\forall\ a \geq a^\star_{t+1} \\ \\ Y^{u0}_{t+1}(a) & ,\forall\ a < a^\star_{t+1}\end{array}\right.\right]$$

Yet, attention must be paid on the fact that if the worker does not find a job and faces human capital depreciation he switches at the next period, either to the type-1 or the type-0 status if its ability is too low. Similarly, we can define $Y^{u1}_t(a)$ and $Y^{u2}_t(a)$ as follows:

$$\begin{aligned} Y^{u1}_t(a) &= b + \beta\left\{\begin{array}{ll} p[\tilde{Y}_{t+1}(a) - \gamma_F] + [1-p]Y^{u1}_{t+1}(a) & ,\forall\ a \geq a^\star_{t+1} \\ \\ p_0\hat{Y}_{t+1}(a) + [1-p_0]Y^{u0}_{t+1}(a) & ,\forall\ a < a^\star_{t+1}\end{array}\right. \\ Y^{u0}_t(a) &= b + \beta\left[p_0\hat{Y}_{t+1}(a) + [1-p_0]Y^{u0}_{t+1}(a)\right] \quad ,\forall\ a < a^\star_t \end{aligned}$$

where the social value of an employed worker with obsolete knowledge is given by:

$$\hat{Y}_t(a) = a + \beta\left[(1-\delta)\hat{Y}_{t+1}(a) + \delta Y^{u0}_{t+1}(a)\right] \quad ,\forall\ a < a^\star_t$$

Therefore, from the planner's point of view it is optimal to train an unemployed worker that faced skill depreciation only if $\tilde{Y}_t(a) - \gamma_f > \hat{Y}_t(a)$. This means that there exists an efficient ability threshold. The latter is denoted $a^\star_t$ and solves $\forall\ t : \tilde{Y}_t(a^\star_t) - \gamma_F = \hat{Y}_t(a^\star_t)$. This implies that the efficient training policy is then characterized by:[5]

$$\begin{aligned} \Delta a^\star_{T-1} &= \gamma_f \\ \\ \Delta a^\star_{T-2} &= \frac{\gamma_f}{\sum_{i=0}^{1}[\beta(1-\delta)]^i} \\ \Delta a^\star_t &= \frac{\gamma_f - \left\{\sum_{i=0}^{T-3-t}\beta\delta[\beta(1-\delta)]^i\left[Y^{u2}_{t+1+i}(a^\star_t) - Y^{u0}_{t++1+i}(a^\star_t)\right]\right\}}{\sum_{i=0}^{T-1-t}[\beta(1-\delta)]^i} \quad ,\ \forall\ t \leq T-3 \end{aligned}$$

This shows (wrt. to the equilibrium policy) that what matters now is the relative social value of training for the unemployed, as defined by $Y^{u2}_t(a) - Y^{u0}_t(a)$, *ie.* the social value

[5] See Appendix D for some details on the derivation.

of having up-to-date knowledge when unemployed with respect to the value of having obsolete knowledge with no perspective to be trained. In particular, this takes into account of the fact that type-2 unemployed workers get probability p to find a job (instead of p_0), and once employed they get a social value $\tilde{Y}_t(a)$ (instead of $\hat{Y}_t(a)$).

2.5 Training externalities

We can now compare both equilibrium and efficient outcomes to highlight training externalities. At this stage, we focus on analytical insights by looking at $t = [T-3, T-2, T-1]$. It is then straightforward to see that equilibrium and optimal training policies are characterized by:

$$\Delta\tilde{a}_{T-1} = \Delta a^{\star}_{T-1} = \gamma_f$$
$$\Delta\tilde{a}_{T-2} = \Delta a^{\star}_{T-2} = \frac{\gamma_f}{\sum_{i=0}^{1}[\beta(1-\delta)]^i}$$

and

$$\begin{aligned}
\Delta a^{\star}_{T-3}\left[\left\{\sum_{i=0}^{2}[\beta(1-\delta)]^i\right\} + \beta^2\delta p\right] &= \gamma_f - (a^{\star}_{T-3} - b)\beta^2\delta(p - p_0) \\
\Delta\tilde{a}_{T-3}\left[\left\{\sum_{i=0}^{2}[\beta(1-\delta)]^i\right\} + \alpha\beta^2\delta p\right] &= \gamma_f - (\tilde{a}_{T-3} - b)\alpha\beta^2\delta(p - p_0)
\end{aligned}$$

It should be first noticed that at the end of the life cycle (from $t \geq T-2$), the equilibrium training policy converges to what is optimal to do. This is quite intuitive because the potential externalities related to training in general human capital falls short when retirement gets closer. On the opposite, for $t = T-3$, we see that the equilibrium training policy is efficient if and only if $\alpha = 1$, *ie.* if the workers get all the bargaining power. Indeed, two types of externalities are not internalized by firms (whereas they are by workers). By inspecting the two equations for $a^{\star}_{T-3}$ and $\tilde{a}_{T-3}$, we see a similar capitalization effect that is determined by $\sum_{i=0}^{1}[\beta(1-\delta)]^i$, but for the equilibrium condition α weights two terms:

- $\beta^2\delta p$ which relates to the poaching externality: general human capital investments can benefit, with some probability, to future employers. Specifically, a worker with up-to-date knowledge fired with probability δ can be re-employed at the next period with probability p.

- $p - p_0$ which relates to the unemployment externality: workers whose ability is high enough for training switch faster from home production to market production (due to a higher job finding probability) and this embodies a social gain that is not valuated by the employers whose bargaining power is $1 - \alpha$.

Therefore, this already emphasizes that externalities related to vocational training are age-dependent so that the optimal policy tool to be implemented should also presumably be. And from this first perspective, this suggests that the size of training externalities is declining at the end of the life cycle. But there is also another issue, that is the moment/age at which the horizon effect on training policies starts to be significant. Obviously, this age should depend on whether training externalities are internalized or not. The next section runs a first quantitative assessment that will address this issue.

3 On the age-dynamics of optimal training subsidies

Overall, we need an empirical investigation to further examine the age-dynamics of training externalities and then look at the optimal age-design of a policy instrument. To that end, our strategy consists in calibrating the model on the french economy to account both for the age-dynamics of training expenditures. In particular, the distance-to-retirement effect on endogenous training decisions is combined with some ad-hoc specifications of job finding probabilities, with a sensitivity analysis with respect to those probabilities. Then, we use the corresponding set of parameters to compute the gap between equilibrium and efficient training policies. Lastly, we determine what would be the optimal subsidy rate of training costs that leads the equilibrium to reach efficiency. This allows to discuss the age-design of this subsidy rate.

3.1 Calibration

The model is calibrated on french data. We use statistics for the life dynamics of vocational training expenditures that are taken from Chéron, Courtioux and Lignon [2015] which refer to the french Labor Force Survey over the period 2003-2010. Total firms' training sponsored expenditures amount approximately to 12 billions euros. The model is simulated at a quarterly frequency, considering $t = [1, 168]$ by referring to workers from 20 to 62 years old. Actually, our quantitative investigation will focus on 30-62 years old workers, hence abstracting from the labor market entry of the youth. Table 1 first reports quite a standard calibration for the discount factor, home production, and the bargaining power of workers. Distribution of abilities follows a Pareto distribution. For γ_f normalized to 1, we choose Δ, that governs the training decision to replicate the average training expenditures as a percentage of wage costs, that is $1,2\%$. The turbulence parameter is taken from Ljungqvist and Sargent [2004], and the corresponding value suggests that in turbulent times the expected unemployment duration before undergoing skill loss is two quarters. We consider a quarterly job destruction probability of 3% per quarter consistent with average duration of employment in France.

Table 1: Model parameters

Parameter	Description	Value
T	Retirement age	168
β	Discount factor	0.99
b	Home production	0.4
α	Bargaining power of workers	0.5
$f(a)$	$1 - b/a$	with $a \geq 0.4$
Δ	Additional output	0.035
π	Turbulence parameter	0.5
γ_f	Training cost	1
δ	Job separation probability	0.03

Lastly, we allow for a robustness analysis with respect to the job finding probabilities, by considering three cases. For each case, we choose p so that the model fits an average employment rate of 80.8% for the 25-54 years old workers, but then consider three ratios for $p0/p$. More precisely, we set: *case 1* (benchmark) $p = 0.185$ and $p_0 = 0.5p$, *case 2* $p_0 = p = 0.13$ which implies that only poaching externality exists, and *case 3* $p = 0.28$ and $p_0 = 0.1p$ which implies that the unemployment externality is magnified.

3.2 Training and employment age-dynamics

Our quantitative assessment starts with computations of equilibrium life cycle dynamics of the ability thresholds and employment rates. Figure 1 shows the results, by considering our benchmark calibration but also the two alternatives cases. In accordance with the data, it comes that workers' access to vocational training is decreasing with age, thereby reflecting the effect of horizon. More specifically, the shorter the distance to retirement, the smaller the payback period and the lower the incentives for firms to train workers. For a retirement age of 62 years, the selection of workers into training programs sharply rises from 50 years old on. Computing related training expenditures by age (in million euros),

Figure 1: Training and employment dynamics

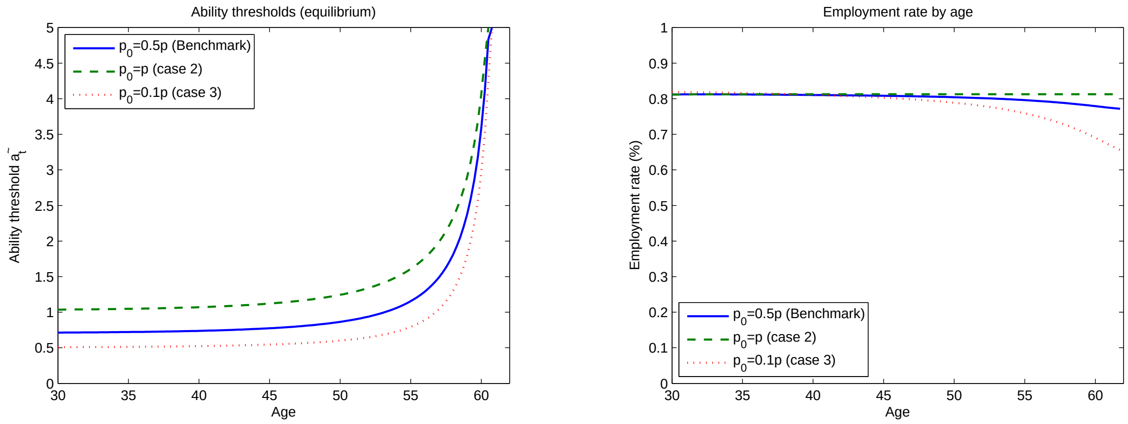


Figure 2: Training expenditure by age

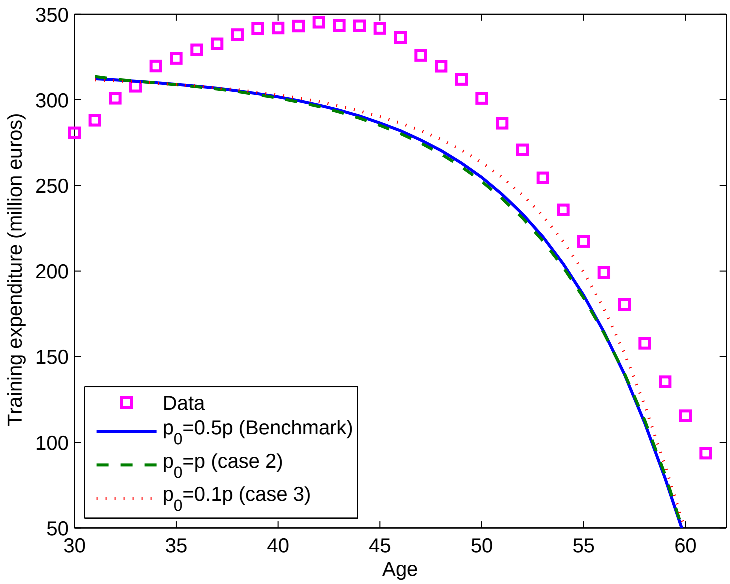


figure 2 further highlights that model's predictions are quite close to what is observed in the data.

We can also notice that training expenditures go down as p_0 gets closer to p (by comparing the three reported cases). The point is indeed that if the gap $p - p_0$ is high, workers with obsolete skills agree with lower wages to access vocational training, because by upgrading their knowledge today they expect that if they turn out to be unemployed in the future their average unemployment spell will be highly reduced. In other words, the willingness of employees to bear part of the cost of training (through wage cuts) enhance firms' incentive to invest in vocational training.

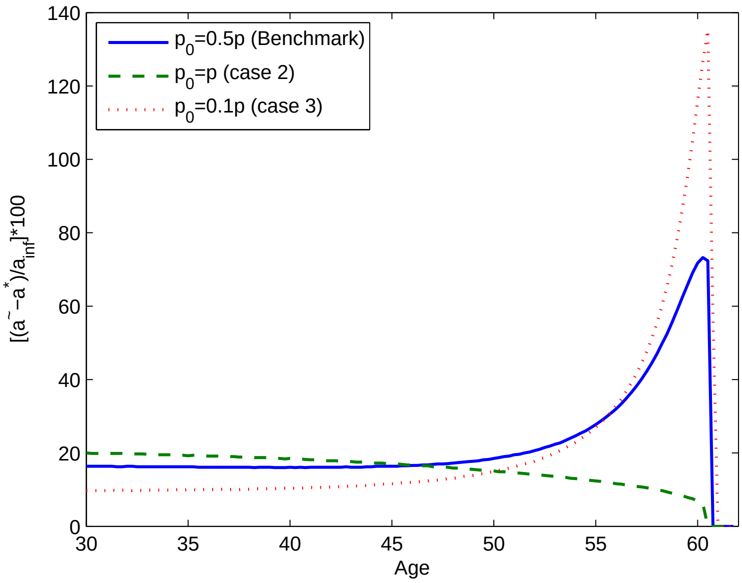
In turn, this age-dynamics of training selection also implies a decrease in employment at the end of the life cycle. More precisely, with our benchmark case, the employment rate falls from 80.8% for the 25-54 years old workers to 78.8%, that is 2 percentage points of employment. In the data, this decrease is greater than 10 percentage points, but obviously this should relate to endogenous age-decreasing job finding probabilities. Moreover, once we consider that type-0 workers face a much more lower job finding probability (case 3), the model is able to account for a 8 percentage points decrease of the employment rate between 25-54 and 55-59.

3.3 The age-design of training externalities

We now propose to compare equilibrium training outcomes with social efficiency. The gap between the two gives a measure of training externalities and their life cycle dynamics. Figure 3 focuses on the gap between ability thresholds, expressed as a percentage of the minimum ability (value of home production). Interestingly, this shows that the age-dynamics is hump shaped, with a sharp rise of this gap over 50 years old. This emphasizes the key role played by externalities for older workers and highlights our main quantitative point. Indeed, we do have a decrease in the size of the externalities at the end of the life cycle (as was emphasized in section 2.5), but we also find that a planner would start to significantly rise selection into training programs later in the life cycle than what firms do in equilibrium. Otherwise stated, if a firm was to take into account externalities, she would delay its increase in ability thresholds, and this means that some older workers that firms do not to train would have to be trained when poaching and unemployment externalities are internalized.

Overall, there exists two opposite forces as worker is aging. On the one hand, firms typically reduce too far from retirement the entry of workers into training programs. But on the other hand, it comes that both private return and social return of training investments collapse to zero when retirement gets closer. This typically requires an hump-shaped age-dynamics of training subsidies to restore social efficiency. As expected, figure 3 shows that this result is strengthened as p_0/p is decreasing, because the role of the unemployment externality is reinforced.

Figure 3: Gap between equilibrium and efficient training policy



3.4 The age-design of optimal training subsidies

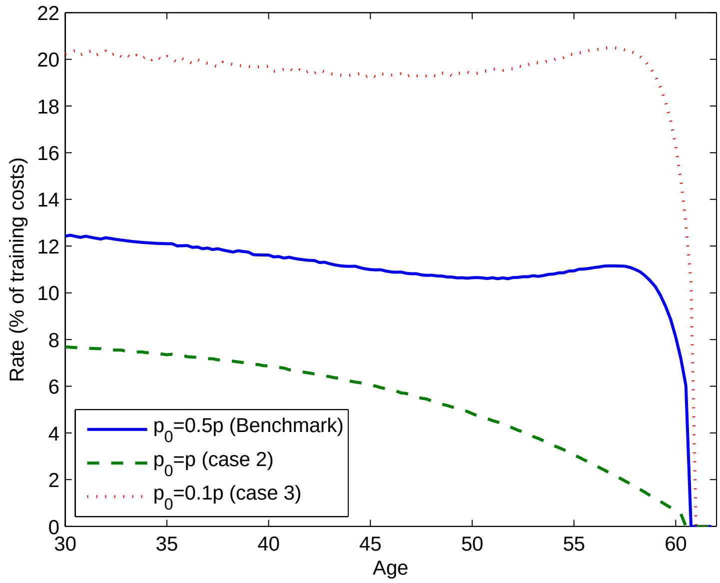
Therefore, we can think about implementing an optimal policy to restore efficiency. We now consider an age-dependent incentive policy which consists in subsidizing a share of the training cost. This subsidy, denoted s_t, is expressed as a percentage of the training cost γ_f; it is derived from following condition:[6]

$$J_{1,t}(a_t^\star) - \gamma_f(1 - s_t) = J_{0,t}(a_t^\star)$$

The age-profile of optimal training subsidies is shown in figure 4. Roughly speaking, this reflects the two opposite forces that govern training externalities as worker is aging. Then, if there is only the poaching externality (*case 2*), the subsidy rate turns out to be strictly decreasing with age, whereas if we do take into account of unemployment externality, a higher subsidy rate is required for workers around 55 years old. Overall, the optimal age-profile of the training subsidy rate is all the more hump-shaped between 45 to 62 that unemployment externality is high. Lastly, this gives also a quantitative measurement of the role played by training externalities, subsidy rates that reach between 15% to 20% of the training costs.

[6]The issue of a fiscal policy to finance those subsidies is not addressed her.

Figure 4: Optimal subsidy rates



4 Conclusion

In this paper, we developed a life cycle model to examine the (in)efficiency of vocational training investments in a frictional labor market context. Our main goal was to examine interactions between investments in human capital and age. We firstly emphasized that poaching and unemployment externalities related to training are age-dependent. Secondly, we argued that this requires an hump-shaped subsidy rate of training costs to restore efficiency. Using a calibration on french data, we finally showed that the optimal policy implies to subsidize training at an increasing rate from 50 years old, but then decreasing at the very end of the life cycle.

That work could be extended in several directions. More particularly, endogenous job creation would not only help to explain the decrease in employment for the older workers, but also to show how sensitive to age is the gap between the job finding probability for workers with up-to-date *versus* obsolete knowledge. It seems important because we already argued that, for probabilities constant across ages, the unemployment externality gets more incidence on the older workers. Obviously, this result should be magnified if the job finding probability of the older workers would be relatively more sensitive to skill obsolescence. This would also allow to examine how age-externalities in the job creation process combine with training externalities. We leave this issue for future research.

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A Workers flows

Let $e_{j,t}(a)$ and $u_{j,t}(a)$ denote the employment and unemployment rates for age t and ability a. The age-dynamic of workers flows is then given by :

For $t = 0$:

Initially, all individuals are endowed with up-to-date skills and enter on the labor market as unemployed of type 2, so that :

- $\forall\ a$:

  $u_{2,t}(a) = f(a)$

For $t = 1$:

The firms now determine at each age an ability threshold $\tilde{a}_t$ above which train a hired worker becomes profitable. We note that some unemployed of type 2 may have undergone a skill loss with a probabilty π : They become unemployed of type 0 if their ability is below the threshold $\tilde{a}_1$, or unemployed of type 1 if their ability is superior or equal to $\tilde{a}_1$.

- $\forall\ a < \tilde{a}_t$,

  $u_{0,t}(a) = f(a)[1-p]\pi$

  $u_{2,t}(a) = f(a)[1-p](1-\pi)$
  $e_{2,t}(a) = f(a)p$

- $\forall \geq \tilde{a}_t$,

  $u_{1,t}(a) = f(a)[1-p]\pi$

  $u_{2,t}(a) = f(a)[1-p](1-\pi)$
  $e_{2,t}(a) = f(a)p$

For $t = 2$:

From this period, we note that some unemployed workers who were able for training in the previous period now have a level of ability too low to be trained : They become unemployed of type 0.

- $\forall\ a < \tilde{a}_{t-1}$:

$$u_{0,t}(a) = u_{0,t-1}(a)[1 - p_0] + u_{2,t-1}(a)[1 - p]\pi$$
$$e_{0,t}(a) = u_{0,t-1}(a)p_0$$

$$u_{2,t}(a) = u_{2,t-1}(a)[1 - p](1 - \pi) + \delta e_{2,t-1}(a)$$
$$e_{2,t}(a) = (1 - \delta)e_{2,t-1}(a) + u_{2,t-1}(a)p$$

- $\forall\ a \in [\tilde{a}_{t-1}; \tilde{a}_t[$:

$$u_{0,t}(a) = u_{1,t-1}(a)[1 - p_0] + u_{2,t-1}(a)[1 - p]\pi$$
$$e_{0,t}(a) = u_{1,t-1}(a)p_0$$

$$u_{2,t}(a) = u_{2,t-1}(a)[1 - p](1 - \pi) + \delta e_{2,t-1}(a)$$
$$e_{2,t}(a) = (1 - \delta)e_{2,t-1}(a) + u_{2,t-1}(a)p$$

- $\forall\ a \geq \tilde{a}_t$:

$$u_{1,t}(a) = u_{1,t-1}(a)[1 - p] + u_{2,t-1}(a)[1 - p]\pi$$
$$e_{1,t}(a) = u_{1,t-1}(a)p$$

$$u_{2,t}(a) = u_{2,t-1}(a)[1 - p](1 - \pi) + \delta e_{2,t-1}(a)$$
$$e_{2,t}(a) = (1 - \delta)e_{2,t-1}(a) + u_{2,t-1}(a)p$$

$\forall\ t \in [3; T-1]$:

From $t = 3$, we get a dynamic of workers flows which breeds until retirement age.

- $\forall\ a < \tilde{a}_{t-1}$,

$$u_{0,t}(a) = u_{0,t-1}(a)[1-p_0] + u_{2,t-1}(a)[1-p]\pi + \delta e_{0,t-1}(a)$$
$$e_{0,t}(a) = (1-\delta)e_{0,t-1}(a) + u_{0,t-1}(a)p_0$$

$$e_{1,t}(a) = (1-\delta)e_{1,t-1}(a)$$

$$u_{2,t}(a) = u_{2,t-1}(a)[1-p](1-\pi) + \delta[e_{1,t-1}(a) + e_{2,t-1}(a)]$$
$$e_{2,t}(a) = (1-\delta)e_{2,t-1}(a) + u_{2,t-1}(a)p$$

- $\forall\ a \in [\tilde{a}_{t-1}; \tilde{a}_t[$,

$$u_{0,t}(a) = u_{1,t-1}(a)[1-p_0] + u_{2,t-1}(a)[1-p]\pi$$
$$e_{0,t}(a) = u_{1,t-1}(a)p_0$$

$$e_{1,t}(a) = (1-\delta)e_{1,t-1}(a)$$

$$u_{2,t}(a) = u_{2,t-1}(a)[1-p](1-\pi) + \delta[e_{1,t-1}(a) + e_{2,t-1}(a)]$$
$$e_{2,t}(a) = (1-\delta)e_{2,t-1}(a) + u_{2,t-1}(a)p$$

- $\forall\ a \geq \tilde{a}_t$,

$$u_{1,t}(a) = u_{1,t-1}(a)[1-p] + u_{2,t-1}(a)[1-p]\pi$$
$$e_{1,t}(a) = u_{1,t-1}(a)p + (1-\delta)e_{1,t-1}(a)$$

$$u_{2,t}(a) = u_{2,t-1}(a)[1-p](1-\pi) + \delta[e_{1,t-1}(a) + e_{2,t-1}(a)]$$
$$e_{2,t}(a) = (1-\delta)e_{2,t-1}(a) + u_{2,t-1}(a)p$$

B Wage bargaining

Let α be the bargaining power of the workers, considered as constant across ages. Wages are solutions of the Nash-sharing rules as follows :

$$\begin{aligned}(1-\alpha)[E_{0,t}(a)-U_{0,t}(a)] &= \alpha J_{0,t}(a)\\(1-\alpha)[E_{1,t}(a)-U_{1,t}(a)] &= \alpha[J_{1,t}(a)-\gamma_f]\\(1-\alpha)[E_{2,t}(a)-U_{2,t}(a)] &= \alpha J_{2,t}(a)\end{aligned}$$

B.1 Type-0 worker, unable for training ($a < \tilde{a}_t$) :

Workers :

$$[E_{0,t}(a)-U_{0,t}(a)] = w_{0,t}(a)-b+\beta(1-\delta)\left[E_{0,t+1}(a)-U_{0,t+1}(a)\right]-\beta p_0\left[E_{0,t+1}(a)-U_{0,t+1}(a)\right]$$

Firms :

$$J_{0,t}(a) = a-w_{0,t}(a)+\beta(1-\delta)J_{0,t+1}(a)$$

According to the sharing rule :

$$(1-\alpha)\left[w_{0,t}(a)-b+\beta(1-\delta)\left[E_{0,t+1}(a)-U_{0,t+1}(a)\right]-\beta p_0\left[E_{0,t+1}(a)-U_{0,t+1}(a)\right]\right]$$
$$=\alpha\left[a-w_{0,t}(a)+\beta(1-\delta)J_{0,t+1}(a)\right]$$

This implies the following wage for a type-0 worker :

$$w_{0,t}(a) = \alpha a+(1-\alpha)b+\alpha\beta p_0 J_{0,t+1}(a)$$

The expected value of a filled job by a type-0 worker is :

$$J_{0,t}(a) = (1-\alpha)(a-b)+\beta(1-\delta)J_{0,t+1}(a)-\alpha\beta p_0 J_{0,t+1}(a)$$

B.2 Type-1 worker, able for training for the last time ($\tilde{a}_t \leq a < \tilde{a}_{t+1}$)

Workers :

$$[E_{1,t}(a) - U_{1,t}(a)] = w_{1,t}(a) - b + \beta(1-\delta)\left[E_{1,t+1}(a) - U_{1,t+1}(a)\right] - \beta p_0\left[E_{0,t+1}(a) - U_{0,t+1}(a)\right] + \beta\delta\left[U_{2,t+1}(a) - U_{0,t+1}(a)\right]$$

Firms :

$$J_{1,t}(a) = (1+\Delta)a - w_{1,t}(a) + \beta(1-\delta)J_{1,t+1}(a)$$

According to the sharing rule :

$$(1-\alpha)\left[w_{1,t}(a) - b + \beta(1-\delta)\left[E_{1,t+1}(a) - U_{1,t+1}(a)\right] - \beta p_0\left[E_{0,t+1}(a) - U_{0,t+1}(a)\right] + \beta\delta\left[U_{2,t+1}(a) - U_{0,t+1}(a)\right]\right] = \alpha\left[(1+\Delta)a - w_{1,t}(a) + \beta(1-\delta)J_{1,t+1}(a) - \gamma_f\right]$$

This implies the following wage for a type-1 worker :

$$\begin{aligned} w_{1,t}(a) &= \alpha\left[(1+\Delta)a - \gamma_f(1-\beta(1-\delta))\right] + (1-\alpha)b + \alpha\beta p_0 J_{0,t+1}(a) \\ &\quad -\beta(1-\alpha)\left[U_{2,t+1}(a) - U_{0,t+1}(a)\right] \end{aligned}$$

The expected value of a filled job by a type-1 worker is :

$$\begin{aligned} J_{1,t}(a) &= (1-\alpha)\left[(1+\Delta)a - b\right] + \beta(1-\delta)J_{1,t+1}(a) - \alpha\beta p_0 J_{0,t+1}(a) \\ &\quad +\alpha\gamma_f(1-\beta(1-\delta)) + \beta\delta(1-\alpha)\left[U_{2,t+1}(a) - U_{0,t+1}(a)\right] \end{aligned}$$

B.3 Type-2 worker, with up-to-date knowledge, who will no longer be able for training if he undergoes a depreciation of human capital ($a < \tilde{a}_{t+1}$) :

Workers :

$$[E_{2,t}(a) - U_{2,t}(a)] = w_{2,t}(a) - b + \beta(1-\delta)\left[E_{2,t+1}(a) - U_{2,t+1}(a)\right] - \beta p\left[E_{2,t+1}(a) - U_{2,t+1}(a)\right]$$
$$+ \beta\pi\left[1-p\right]\left[U_{2,t+1}(a) - U_{0,t+1}(a)\right]$$

Firms :

$$J_{2,t}(a) = (1+\Delta)a - w_{2,t}(a) + \beta(1-\delta)J_{2,t+1}(a)$$

According to the sharing rule :

$$(1-\alpha)\left[w_{2,t}(a) - b + \beta(1-\delta)\left[E_{2,t+1}(a) - U_{2,t+1}(a)\right] - \beta p\left[E_{2,t+1}(a) - U_{2,t+1}(a)\right]\right.$$
$$\left.+\beta\pi\left[1-p\right]\left[U_{2,t+1}(a) - U_{0,t+1}(a)\right]\right] = \alpha\left[(1+\Delta)a - w_{2,t}(a) + \beta(1-\delta)J_{2,t+1}(a)\right]$$

This implies the following wage for a type-2 worker :

$$w_{2,t}(a) = \alpha(1+\Delta)a + (1-\alpha)b + \alpha\beta p J_{2,t+1}(a) - \beta\pi(1-\alpha)\left[1-p\right]\left[U_{2,t+1}(a) - U_{0,t+1}(a)\right]$$

The expected value of a filled job by a type-2 worker is :

$$\begin{aligned} J_{2,t}(a) &= (1-\alpha)\left[(1+\Delta)a - b\right] + \beta(1-\delta)J_{2,t+1}(a) - \alpha\beta p J_{2,t+1}(a) \\ &\quad +\beta\pi(1-\alpha)\left[1-p\right]\left[U_{2,t+1}(a) - U_{0,t+1}(a)\right] \end{aligned}$$

C Equilibrium training

The training policy of the firms consist in choosing $\tilde{a}_t \; \forall \; t$ so that :

$$J_{1,t}(\tilde{a}_t) - \gamma_f = J_{0,t}(\tilde{a}_t)$$

The equilibrium ability threshold solves :

$$(1-\alpha)\left[(1+\Delta)\tilde{a}_t - b\right] + \beta(1-\delta)J_{1,t+1}(\tilde{a}_t) - \alpha\beta p_0 J_{0,t+1}(\tilde{a}_t) + \alpha\gamma_f(1-\beta(1-\delta))$$
$$+\beta\delta(1-\alpha)\left[U_{2,t+1}(\tilde{a}_t) - U_{0,t+1}(\tilde{a}_t)\right] = (1-\alpha)(\tilde{a}_t - b) + \beta(1-\delta)J_{0,t+1}(\tilde{a}_t) - \alpha\beta p_0 J_{0,t+1}(\tilde{a}_t)$$

We get the following condition :

$$\Delta\tilde{a}_t \;\; = \;\; \gamma_f - \frac{\beta(1-\delta)}{1-\alpha}\left[(J_{1,t+1}(\tilde{a}_t) - \alpha\gamma_f) - J_{0,t+1}(\tilde{a}_t)\right] - \beta\delta\left[U_{2,t+1}(\tilde{a}_t) - U_{0,t+1}(\tilde{a}_t)\right]$$

Using the equations of expected values of filled jobs and the wage equations, we can demonstrate that :

$$\left[(J_{1,t+1}(\tilde{a}_t) - \alpha\gamma_f) - J_{0,t+1}(\tilde{a}_t)\right] = (1-\alpha)\Delta\tilde{a}_t + \beta(1-\delta)\left[(J_{1,t+2}(\tilde{a}_t) - \alpha\gamma_f) - J_{0,t+2}(\tilde{a}_t)\right]$$
$$+\beta\delta(1-\alpha)\left[U_{2,t+2}(\tilde{a}_t) - U_{0,t+2}(\tilde{a}_t)\right]$$

$$U_{2,t+1}(\tilde{a}_t) - U_{0,t+1}(\tilde{a}_t) = \frac{\alpha\beta}{1-\alpha}\left[pJ_{2,t+2}(\tilde{a}_t) - p_0 J_{0,t+2}(\tilde{a}_t)\right] + \beta[1-\pi(1-p)]\left[U_{2,t+2}(\tilde{a}_t) - U_{0,t+2}(\tilde{a}_t)\right]$$

We can demonstrate recursively that :

$$\Delta\tilde{a}_t \;\; = \;\; \frac{\gamma_f - \sum_{i=0}^{T-3-t}\beta\delta\left[\beta(1-\delta)\right]^i\left[U_{2,t+1+i}(\tilde{a}_t) - U_{0,t+1+i}(\tilde{a}_t)\right]}{\sum_{i=0}^{T-1-t}\left[\beta(1-\delta)\right]^i}$$

Then, we can determine in particular the following ability thresholds :

$$\Delta\tilde{a}_{T-1} = \gamma_f$$

$$\Delta\tilde{a}_{T-2}\left[\left\{\sum_{i=0}^{1}\left[\beta(1-\delta)\right]^i\right\}\right] = \gamma_f$$

$$\Delta\tilde{a}_{T-3}\left[\left\{\sum_{i=0}^{2}\left[\beta(1-\delta)\right]^i\right\} + \alpha\beta^2\delta p\right] = \gamma_f - (\tilde{a}_{T-3} - b)\alpha\beta^2\delta\left[p - p_0\right]$$

D Efficient training

The efficient training policy, based on social value of workers, consist in choosing the optimal ability threshold $a_t^\star$ which satisfies $\forall\ t:\ \tilde{Y}_t(a_t^\star) - \gamma_F = \hat{Y}_t(a_t^\star)$.

The efficient ability threshold solves :

$$(1+\Delta)a_t^\star + \beta\left[(1-\delta)\tilde{Y}_{t+1}(a_t^\star) + \delta Y_{t+1}^{u2}(a_t^\star)\right] - \gamma_f = a_t^\star + \beta\left[(1-\delta)\hat{Y}_{t+1}(a_t^\star) + \delta Y_{t+1}^{u0}(a_t^\star)\right]$$

We get the following condition :

$$\begin{aligned} \Delta a_t^\star &= \gamma_f - \beta(1-\delta)\left[\tilde{Y}_{t+1}(a_t^\star) - \hat{Y}_{t+1}(a_t^\star)\right] - \beta\delta\left[Y_{t+1}^{u2}(a_t^\star) - Y_{t+1}^{u0}(a_t^\star)\right] \\ &= \gamma_f - \beta(1-\delta)\left[\left(\tilde{Y}_{t+1}(a_t^\star) - Y_{t+1}^{u2}(a_t^\star)\right) - \left(\hat{Y}_{t+1}(a_t^\star) - Y_{t+1}^{u0}(a_t^\star)\right)\right] \\ &\quad -\beta\left[Y_{t+1}^{u2}(a_t^\star) - Y_{t+1}^{u0}(a_t^\star)\right] \end{aligned}$$

Using the equations of social values, we can demonstrate that :

$$\begin{aligned} &\left(\tilde{Y}_{t+1}(a_t^\star) - Y_{t+1}^{u2}(a_t^\star)\right) - \left(\hat{Y}_{t+1}(a_t^\star) - Y_{t+1}^{u0}(a_t^\star)\right) \\ &= \Delta a_t^\star + \beta(1-\delta)\left[\left(\tilde{Y}_{t+2}(a_t^\star) - Y_{t+2}^{u2}(a_t^\star)\right) - \left(\hat{Y}_{t+2}(a_t^\star) - Y_{t+2}^{u0}(a_t^\star)\right)\right] \\ &\quad - \beta\left[p\left(\tilde{Y}_{t+2}(a_t^\star) - Y_{t+2}^{u2}(a_t^\star)\right) - p_0\left(\hat{Y}_{t+2}(a_t^\star) - Y_{t+2}^{u0}(a_t^\star)\right)\right] + \beta\pi\left[1-p\right]\left[Y_{t+2}^{u2}(a_t^\star) - Y_{t+2}^{u0}(a_t^\star)\right] \end{aligned}$$

We can also show that :

$$\begin{aligned} Y_{t+1}^{u2}(a_t^\star) - Y_{t+1}^{u0}(a_t^\star) &= \beta\left[p\left(\tilde{Y}_{t+2}(a_t^\star) - Y_{t+2}^{u2}(a_t^\star)\right) - p_0\left(\hat{Y}_{t+2}(a_t^\star) - Y_{t+2}^{u0}(a_t^\star)\right)\right] \\ &\quad +\beta\left[Y_{t+2}^{u2}(a_t^\star) - Y_{t+2}^{u0}(a_t^\star)\right] - \beta\pi\left[1-p\right]\left[Y_{t+2}^{u2}(a_t^\star) - Y_{t+2}^{u0}(a_t^\star)\right] \end{aligned}$$

It was noted that :

$$\begin{aligned} &\left(\tilde{Y}_{t+1}(a_t^\star) - Y_{t+1}^{u2}(a_t^\star)\right) - \left(\hat{Y}_{t+1}(a_t^\star) - Y_{t+1}^{u0}(a_t^\star)\right) \\ &= \Delta a_t^\star + \beta(1-\delta)\left[\left(\tilde{Y}_{t+2}(a_t^\star) - Y_{t+2}^{u2}(a_t^\star)\right) - \left(\hat{Y}_{t+2}(a_t^\star) - Y_{t+2}^{u0}(a_t^\star)\right)\right] \\ &\quad - \left[Y_{t+1}^{u2}(a_t^\star) - Y_{t+1}^{u0}(a_t^\star)\right] + \beta\left[Y_{t+2}^{u2}(a_t^\star) - Y_{t+2}^{u0}(a_t^\star)\right] \end{aligned}$$

Using the same method as in equilibrium, we get the following condition :

$$\Delta a_t^\star = \frac{\gamma_f - \left\{\sum_{i=0}^{T-3-t} \beta\delta \left[\beta(1-\delta)\right]^i \left[Y_{t+1+i}^{u2}(a_t^\star) - Y_{t++1+i}^{u0}(a_t^\star)\right]\right\}}{\sum_{i=0}^{T-1-t} \left[\beta(1-\delta)\right]^i}$$

Then, we can determine the following ability thresholds :

$$\Delta a_{T-1}^\star = \gamma_f$$

$$\Delta a_{T-2}^\star \left[\left\{\sum_{i=0}^{1} \left[\beta(1-\delta)\right]^i\right\}\right] = \gamma_f$$

$$\Delta a_{T-3}^\star \left[\left\{\sum_{i=0}^{2} \left[\beta(1-\delta)\right]^i\right\} + \beta^2\delta p\right] = \gamma_f - (a_{T-3}^\star - b)\beta^2\delta \left[p - p_0\right]$$