Contributing to public defense in a contest model

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Abstract

I augment the standard Tullock contest by adding a first stage in which each of the potential contestants has the option of contributing some resources to a public defender or government. In the subsequent subgame, if one of the contestants attacks the other, then the government contributes its resources to the defence of the agent that is attacked. I show that, if the resource distribution is not too unequal, agents make positive contributions to government in equilibrium and there is no fighting. The deterrence equilibria are pareto superior to the corresponding equilibria of the pure Tullock contest. The Rawlsian criterion yields the most efficient equilibrium for each given resource distribution, hence progressive taxation is efficient in this model. Finally, for a range of very unequal resource distributions, the equilibrium size of government is too large.

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1. Introduction

The question of how stable property rights emerge out of a state of anarchy has exercised social thinkers from the very earliest times. In contemporary economic literature, a construct that has been widely used to investigate this question is the rational contest model, in which agents can use resources in their possession to engage in production, or to wrest away resources from other agents. The contest model is similar to constructs that been used to analyse lobbying contests and patent races. Consequences of verious formulations of the nature of contest and the form of the contest success function are explored in a number of papers by Hirshleifer (1991) Hirshleifer (1995), Skaperdas (1992), Grossman and Kim (1995), as well as several others.

Many of these papers investigate conditions under which, in the absence of an external enforcer, the potential contestants will enter into active conflict, and conditions under which they will coexist in peace. In Hirshleifer's formulation resources that are devoted to conflict can be used both for aggression and defence, thus an investment to dissuade the adversary may also tun out to provide incentive for aggression. Grossman and Kim consider investments that are earmarked for aggression (e.g., cannons) or defence (e.g., fortification) and obtain equilibria in which peace may sometimes prevail.

One conclusion that emerges from most of these models is that conflict is more likely when there is high inequality between the agents, and in these cases the poorer agent is more likely to be the aggressor.

Surprisingly, however, very few contributions explore the possibility that the potential contestants, in anticipation of the possible destructiveness of conflict, may enter into cooperation to create an enforcement mechanism as a public good. An exception is McBride, Milante, and Skaperdas (2011), who explore a model in which contestants can invest in a state, which is able to protect from conflict a fraction of all resources; the fraction being determined by the total investment (see also McBride and Skaperdas, 2007).

In this paper I use a simpler construction. As in McBride, Milante, and Skaperdas (2011) the two potential contestants choose to make contributions to enable a public defender. In the subsequent subgame each contestant has a choice to attack the other. If one of the contestants chooses to be an aggressor (and the other does not), then the public defender contributes its resources to the defence of the victim.

I find that peace prevails (though at a cost) except when inequality is extreme, in which case agents no longer contribute to public defence in equilibrium. For a large range of parameter values there are multiple equilibria, with the richer agent contributing a larger or smaller fraction of the public defence. With appropriate investments, peace becomes incentive compatible for two reasons; first, resources invested in public defence are no longer available as conflict payoffs to the contestants, making conflict less attractive, and secondly the same defence investment acts as a deterrant against aggression by both contestants.

Two additional results are of interest. First, when there are multiple equilibria, the most efficient equilibrium is always the one in which the richer agent makes the largest contribution consistent with equilibrium. If we interpret these contributions as taxes determined by a participatory government, then the efficient taxation scheme is the most progressive scheme that is consistent with peace. Secondly, we find that there is a range where inequality is high (but not sufficiently extreme for government to break down) where a contest would in fact be more efficient than a peace equilibrium. An interpretation is that, when inequality is high, government is inefficiently large.

Beviá and Corchón (2010), which is in some ways close to this paper, consider the possibility that the richer agent may transfer some of her wealth to the poorer in order to avoid conflict. Such transfers reduce inequality and therefore the likelihood of conflict. However, when we introduce this option in the present model, we find that contributions to public defence is more attractive to the richer agent than transfers to the poorer agent.

The next section lays out the canonical contest model in its simplest form, and derives the equilibrium outcome. Section 3 describes the model with investment in public defence. Section 4 establishes the equilibria. Section 5 discusses efficiency concerns and identifies the most efficient equilibria. It also shows that the worst peace equilibria are more efficient than the pure contest outcome. The main results are summarised in this section. Section 6 concludes.

2. Background: pure contest

2.1. Setting

I adopt a simple version of the standard model (e.g., Hirschleifer). This section describes the base model and its equilibria. Modifications are introduced later.

There is one unit of resources distributed between two agents, 1 and 2. Without loss of generality we assume that agent 1 is the less rich agent.

$$R_1 + R_2 = 1, \qquad 0 < R_1 \leq R_2$$

Each agent $i = 1, 2$ can devote some or all of his resources $x_i \leq R_i$ as arms to fight. Investments are made simultaneously.

If at least one agent chooses to fight (or attack) then they fight. If they fight then the remaining resources are redistributed between the agents in proportion with their arms. Alternatively, each agent succeeds in capturing the entire remaining resources with a probability equal to his share of the total arms. Agents are risk-neutral, so the two above interpretations are equivalent. Agent i's payoff is Π^i.

$$\Pi_i(R, x, \text{war}) \;=\; \frac{x_i}{x_i + x_j}[1 - (x_i + x_j)]$$

If neither agent chooses to fight then each retains his remaining resources

$$\Pi_i(R, x, \text{peace}) \;=\; R_i - x_i$$

.

Each agent maximizes his payoff Π_i.

2.2. Solution

SPNE is the natural solution concept. In the last stage, i will attack if

$$\Pi_i(R, x, \text{war}) \ > \ \Pi_i(R, x, \text{peace}) \quad \Rightarrow \frac{x_i}{R_i} > \frac{x_j}{R_j}$$

If i attacks, he will choose x_i to maximize

$$\max_{x_i} \Pi_i^{\text{war}} \quad \Rightarrow \quad x_i = \min\{\sqrt{x_j} - x_j, R_i\}$$

Similarly, to defend j will choose $x_j = \min\{\sqrt{x_i} - x_i, R_j\}$

Note that $\sqrt{x_j} - x_j$ reaches a maximum of $\frac{1}{4}$ when $x_j = \frac{1}{4}$, which is also a fixed point of $y = \sqrt{x} - x$. The agents' response functions (x_i as a function of x_j) are plotted in Figure 1.

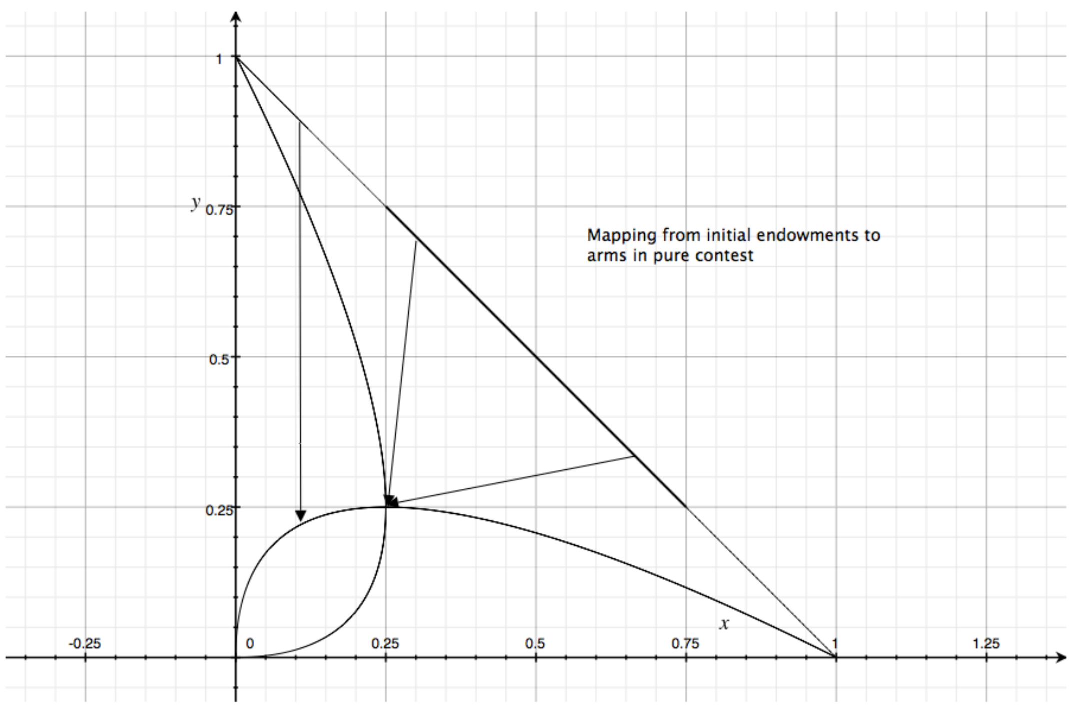

Figure 1: Optimal attack and defence in pure contest

First, note that in equilibrium each player invests positive amounts in arms. The equilibrium investments and payoffs are:

If $\frac{1}{4} \leq R_1 \leq R_2$, then investments are $x_1 = x_2 = \frac{1}{4}$ and payoffs are $\Pi_1 = \Pi_2 = \frac{1}{4}$.

If $R_1 < \frac{1}{4} < R_2$ then investments are $x_1 = R_1$, $x_2 = \sqrt{R_1} - R_1$, and payoffs are $\Pi_1 = \sqrt{R_1}(1 - \sqrt{R_1})$, $\Pi_2 = (1 - \sqrt{R_1})^2$.

There is war except in the case $R_1 = R_2$. When $R_1 = R_2$, the contestants arm optimally in equilibrium, and subsequently are indifferent between war and peace.

We will denote the equilibrium outcomes of pure contest by the superscript C, i.e., $x_1^C, x_2^C, \Pi_1^C, \Pi_2^C$.

The equilibrium payoffs of an agent are plotted against his (share of the) intital endowment in Figure 2.

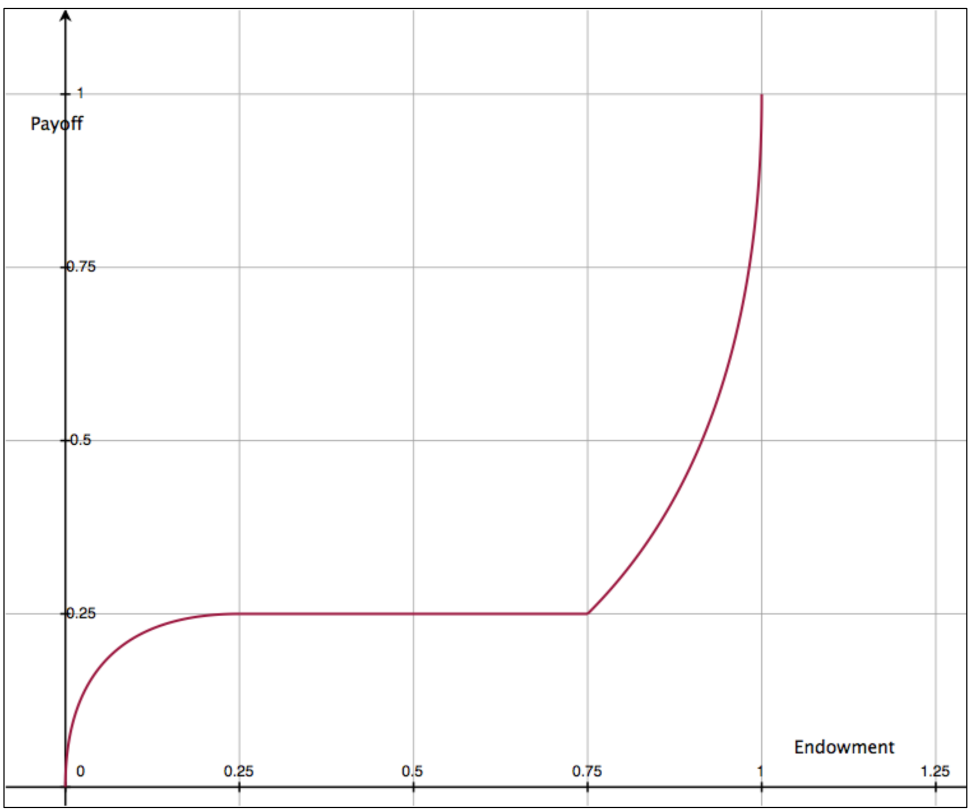

Figure 2: Contest payoffs plotted against endowment

3. Investing in public defence

We augment this game by adding a first decision stage before the players choose their investments in arms.

- First, each player simultaneously chooses to invest an amount g_i to endow a public defender ("government").
- Next investments in private arms are chosen.
- Finally, attack decisions are made.
- If both attack, then the government stands aside. A pure contest occurs using only private arms to divide the remaining resources.
- If neither attacks then there is peace and each consumes his remaining resources.
- However, if agent i chooses to attack and agent j does not, then the government adds its resources to the defence of j. Payoffs are as before.

3.1. The game

We start with $R_1 + R_2 = 1 \quad 0 < R_1 \leq R_2$. The game has three stages.

3.1.1. Game form

Stage 1 (game Γ): Agents simultaneously choose the amount g_i each will contribute to public defence, subject to $g_i \leq R_i$.

- A pair (g_1, g_2) is a *contribution profile* (or *contribution*).
- Let $g = g_1 + g_2$ and $\mathbf{g} = (g_1, g_2)$.
- Define $w_i \;=\; R_i - g_i$, and $\mathbf{w} = (w_1, w_2)$.

Stage 2 (subgame Γ_2): Agents observe $\mathbf{g}$ and simultaneously choose their arms investments $x_i \leq w_i$.

- A pair (x_1, x_2) is an *arms profile* (or *arms*).
- Let $x = x_1 + x_2$, and $\mathbf{x} = (x_1, x_2)$.

Stage 3 (subgame Γ_3): Agents observe $\mathbf{x}$. Then they simultaneously choose $a_i \in \{0, 1\}$. [0 is "defend", 1 is "attack".]

- A pair (a_1, a_2) is an *attack profile*. Let $\mathbf{a} = (a_1, a_2)$.

We use $z = [\mathbf{g}, \mathbf{x}, \mathbf{a}]$ to denote the sequence of decisions in a play of the game.

3.1.2. Payoffs and equilibria

If $(a_1, a_2) = (0, 0)$, then

$$\Pi_i(z) = R_i - g_i - x_i, \quad i = 1, 2.$$

If $(a_1, a_2) = (1,1)$ then

$$\Pi_i(z) = \frac{x_i}{x_i + x_j}[1 - x - g]$$

If $a_i = 1$ and $a_j = 0$, then

$$\begin{aligned}\Pi_i(z) &= \frac{x_i}{x_i + x_j + g}[1 - x - g] \\ \Pi_j(z) &= \frac{x_j + g}{x_i + x_j + g}[1 - x - g]\end{aligned}$$

$z = [\mathbf{g}, \mathbf{x}, \mathbf{a}]$ is an *equilibrium* if it is a subgame-perfect Nash equilibria of the game Γ.

3.2. Aggression and deterrence

We will be particularly interested in equilibria in which both players choose not to arm, and hence not to attack. From the discussion of the pure contest model, it is obvious that this requires positive contributions to public defence by at least one player. Further, the sum of the contributions must be large enough that each player prefers to not arm and not attack given that the opponent also does not arm.

Definitions: We say that a player i is *deterred* by a contribution profile $\mathbf{g} = (g_1, g_2)$ if, following $\mathbf{g}$, i finds it optimal to not arm (and hence not attack) even when the opponent does not arm, i.e., $x_j = 0$. Correspondingly $\mathbf{g}$ is *full deterrent* if both players are deterred in Γ_2 following $\mathbf{g}$. Finally, $\mathbf{g}$ is *minimal full deterrent* if there does not exist $\mathbf{g}' \lneqq \mathbf{g}$ which is also full-deterrent.

Lemma 1 *A contribution profile* $\mathbf{g}$ *is a full deterrent if* $g \geq \hat{g}(\mathbf{w})$, *where*

$$\hat{g}(\mathbf{w}) = \begin{cases} (1 - \sqrt{\min\{w_1, w_2\}})^2 & \text{if} \quad \min\{w_1, w_2\} \geq \frac{1}{4} \\ \dfrac{1}{2} - \min\{w_1, w_2\} & \text{if} \quad \min\{w_1, w_2\} < \frac{1}{4} \end{cases}$$

[All proofs are in the appendix]

Note that to ensure full deterrence it is sufficient to deter the player who has the smaller remaining resource endowment $\min\{w_1, w_2\}$ after contributions. Further, the minimum contribution needed for full deterrence increases as $\min\{w_1, w_2\}$ falls.

Lemma 2 *(i) Let* $g > 0$*, then in the equilibrium of the subgame* Γ_2 *we must have* $\mathbf{a}^* \neq (1,1)$.

(ii) If z^* *is an equilibrium outcome with* $a_i^* = 1$*, then* $g_i^* = 0$.

Lemma 2 is self-evident (though a proof is provided). It says that, first, if a positive contribution has been made then both players will not attack in equilibrium. If one player attacks, then the other is better off not attacking since he gets the benefit of the public defence. Secondly, if a player attacks in equilibrium, then he will not contribute to public defence, since the defence takes away from his resources, and reduces the expected payoff from attacking.

It then follows that, in equilibrium, either the total contribution is full deterrent and neither player invests in private arms, or there is a contest in which case at most one player (who does not attack) contributes to public defence. If he does so, he contributes an amount not exceeding his optimal investment in arms, and then further invests in private arms to make up the remainder of his optimal defence. Thus in the ensuing conflict, each player actually has at his disposal an amount of arms that equals his optimal arms investment in pure contest.

Thus in an equilibrium with conflict, if an agent invests in public defence that investment is inconsequential, and the agent is indifferent between directing those resources to public defence or private arms. To avoid unnecessary complication, we will assume that in such a case the agent regrains from public investment.

Assumption 1 *In an equilibrium z^* if there is war and an agent is indifferent between contributing an amount g_j to public defence or adding it to his private arms, then he adds it to his private arms.*

Proposition 1 *If z^* is an equilibrium outcome, then either (i) g^* is minimal full deterrent with $\mathbf{x}^* = (0,0)$ and $\mathbf{a}^* = (0,0)$, or (ii) $\Pi(R, z^*) = \Pi^C(R, x^C, war)$.*

4. Deterrence equilibria

From Proposition 1 and Assumption 1 it follows that in equilibrium, agents will either together contribute enough to ensure full deterrence, or they will not invest in public defence at all. In the former case, we must have $g = \hat{g}(\mathbf{w})$, the minimum contribution required for full deterrence. Additional contribution is costly to the contributor and does not produce additional payoff.

By Lemma 1 the minimum full-deterrence contribution g is uniquely determined by the smaller of the two remaining resource endowments. Hence we can identify the vectors $\mathbf{w}$ that are compatible with full deterrence.

Consider the subgame Γ_2 with initial post-contribution allocation (w_1, w_2) and associated total public defence contribution $g = 1 - (w_1 + w_2)$. W.l.o.g. let $w_1 = \min\{w_1, w_2\}$.

Proposition 2 *Suppose the contributions (g_1, g_2) in the first stage are such that:*

$$w_2 \leq \begin{cases} \frac{1}{2} & \text{if} \quad w_1 < \frac{1}{4} \\ 2(\sqrt{w_1} - w_1) & \text{if} \quad w_1 \in [\frac{1}{4}, \frac{4}{9}] \end{cases} \tag{1}$$

Then the equilibrium in the subgame Γ_2 is $(\mathbf{x}, \mathbf{a}) = (0,0)$, i.e., peace with no expenditure on private arms.

There are no peace equilibria in subgame Γ_2 when $\min\{w_1, w_2\} > \frac{4}{9}$.

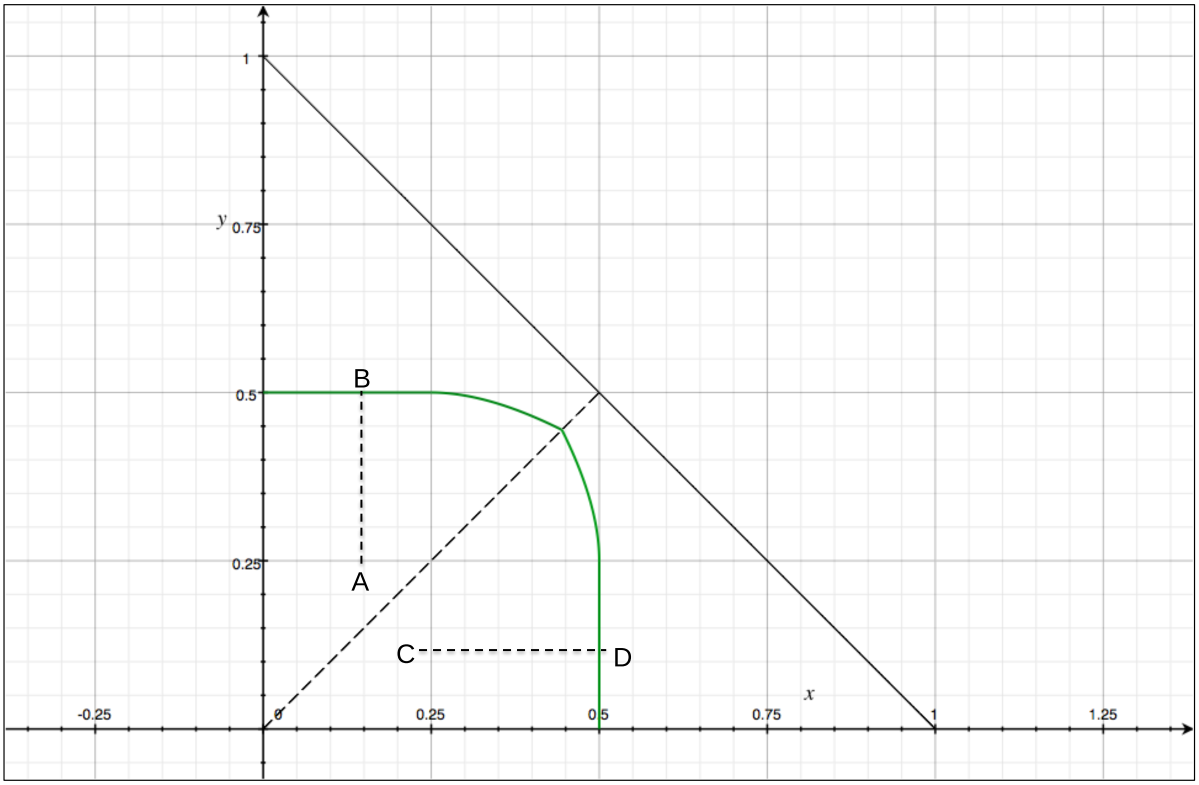

Figure 3: Payoff frontier with full deterrence

For any value of w_1, we can find the largest value of w_2 that is consistent with full-deterrence using Proposition 2. This defines the full-deterrence frontier, mapped in Figure 3.

Figure 3 shows the consumption pairs that are attainable with full deterrence. The line joining $(1,0)$ and $(0,1)$ plots the possible distributions of initial resources. We restrict attention to the section of this line lying above the 45-degree line, where $R_1 \leq R_2$. The analysis of the complementary segment is symmetrical.

The curved frontier is the limit of the consumption pairs (w_1, w_2) that are consistent with full deterrence.[1] To see that allocation below the frontier also induce full-deterrence, note that in an allocation such as A, the public contribution is larger than in B, but $\min\{w_1, w_2\} = w_1$ is unchanged. Thus since B is compatible with full-deterrence so is A. A similar argument applies to C relative to D.

[1]Since full-deterrence implies $\mathbf{x} = 0$ in the subgame, w_i is indeed the consumption of i in the equilibrium of the subgame.

4.1. Deterrence contributions

For a given resource allocation $\mathbf{R}$, consider the set of minimal full deterrence contributions $G(\mathbf{R})$. If $\mathbf{g} \in G(\mathbf{R})$, then it follows that $\mathbf{w} = \mathbf{R} - \mathbf{g}$ is a consumption pair on the full deterrence frontier. Let $W(\mathbf{R})$ be the set of such consumption vectors. Note that no vector in $W(\mathbf{R})$ (weakly) dominates any other vector in $W(\mathbf{R})$.

In figure 3, if $\mathbf{R} \gg (\frac{1}{4}, \frac{1}{4})$, then $W(\mathbf{R})$ is the segment of the full deterrence frontier contained in the rectangle defined by $\mathbf{R}$ and $(\frac{1}{4}, \frac{1}{4})$. If $R_1 \leq \frac{1}{4}$, then $W(\mathbf{R}) = \{(R_1, \dfrac{1}{2})\}$, and if $R_2 \leq \frac{1}{4}$ then $W(\mathbf{R}) = \{(\dfrac{1}{2}, R_2)\}$.

Thus when R_1 or R_2 is $\leq \frac{1}{4}$ the full deterrence consumption vector is unique, but when $R_i \in (\frac{1}{4}, \frac{3}{4}), i = 1, 2$, there is a continuum of such vectors. Using Proposition 2, we can define the range of feasible consumptions under full deterrence for an individual player.

Proposition 3 *Let $Z(\mathbf{R})$ be the set of minimal full-deterrence outcomes corresponding to initial resource allocation $\mathbf{R}$. Then the set of attainable consumptions for Player i in outcome $z \in Z(\mathbf{R})$ are (where $j \neq i$):*

$$w_i \begin{cases} = R_i & \text{if} \quad R_i \leq \frac{1}{4} \\ \in [\frac{1}{4}, R_i] & \text{if} \quad R_i \in (\frac{1}{4}, \frac{1}{2}] \\ \in [\frac{1}{2}\{1 + \sqrt{(2R_j - 1)}\}, \frac{1}{2}] & \text{if} \quad R_i \in (\frac{1}{2}, \frac{5}{9}] \\ \in [2\{\sqrt{1 - R_j} - (1 - R_2)\}, \frac{1}{2}] & \text{if} \quad R_i \in (\frac{5}{9}, \frac{3}{4}] \\ = \frac{1}{2} & \text{if} \quad R_i > \frac{3}{4} \end{cases} \tag{2}$$

In figure 4 we plot the lower and upper bounds for the payoffs for player i that remain after contributions that are compatible with minimal full deterrence (with complementary contributions by player j), corresponding to each endowment of resources. Note that the curvature of the full deterrence frontier in the range $R_1 \in (\frac{1}{4}, \frac{1}{2})$ implies that the contributions of the two players are imperfect substitutes; a reduction in g_2 must be compensated by a more than equal increase in g_1. The reverse is true in the range $R_1 \in (\frac{1}{2}, \frac{3}{4})$.

5. Equilibria and efficiency

Proposition 3 describes the contribution profiles that are candidates for full deterrence equilibria. Observe that the richer player must always contribute to a full deterrence outcome. The poorer player may not contribute, and will indeed not contribute at all when his initial resource endowment is less than $\frac{1}{4}$. In order for full deterrence to be an equilibrium outcome, it is necessary that each player that contributes has a payoff under full deterrence that is no less than the payoff he would obtain under pure contest.

Figure 5 superimposes the full deterrence payoffs (Figure 4) on the pure contest payoffs (Figure 2) for a given player. The pure contest payoffs are strictly greater than

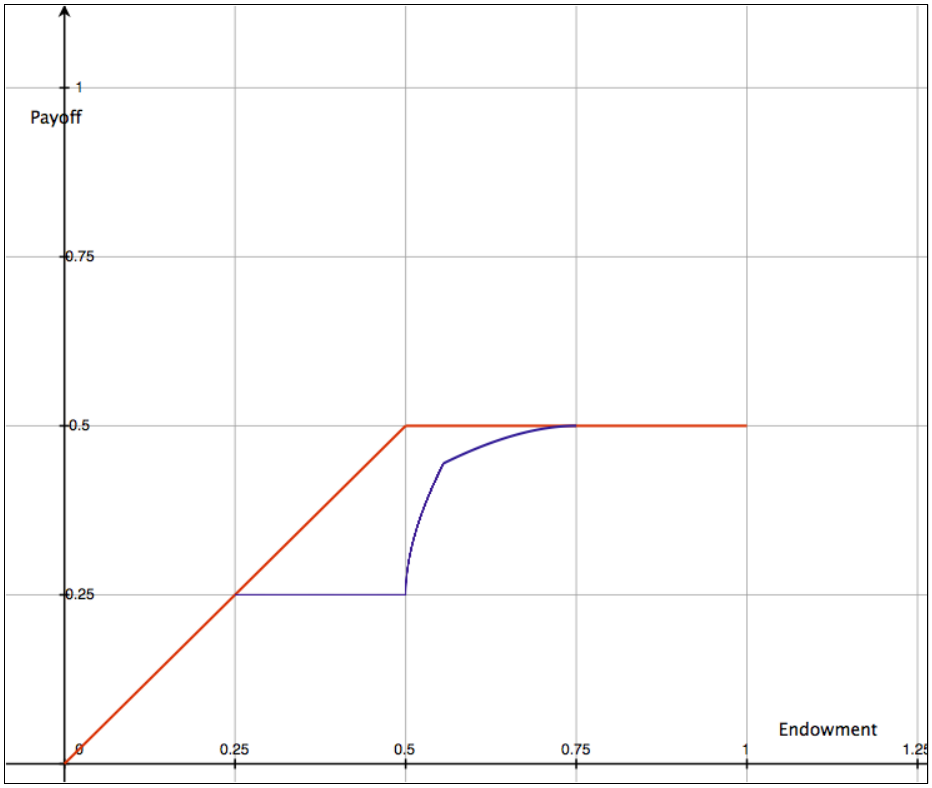

Figure 4: Maximum and minimum payoffs with full deterrence

full deterrence payoffs for $R_i \in (0, \frac{1}{4})$, and in $R_i \in (\sqrt{2} - \dfrac{1}{2}, 1]$. In the former range, player i is the poorer player and does not contribute to public defence in equilibrium, hence he cannot decide on full deterrence. But in the upper range, it is the richer player that makes the entire contribution, hence the choice between conflict and deterrence is his to make. It follows that if $\max\{R_1, R_2\} \in (\sqrt{2} - \dfrac{1}{2}, 1]$, then the richer player will not invest in deterrence, and the equilibrium outcome will be pure conflict. For $\max\{R_1, R_2\} \in (\dfrac{1}{2}, \sqrt{2} - \dfrac{1}{2}]$, on the other hand, deterrence is weakly preferred if the richer player makes the maximum contribution, and strictly preferred if the poorer player makes any contribution at all, hence full deterrence is the equilibrium outcome.

This establishes the equilibria corresponding to the different resource endowments, summarised in the following proposition.

Proposition 4 *If $R_1, R_2 \in [\frac{3}{2} - \sqrt{2}, \sqrt{2} - \dfrac{1}{2}]$ and $R_1 \neq R_2$, then all equilibria are*

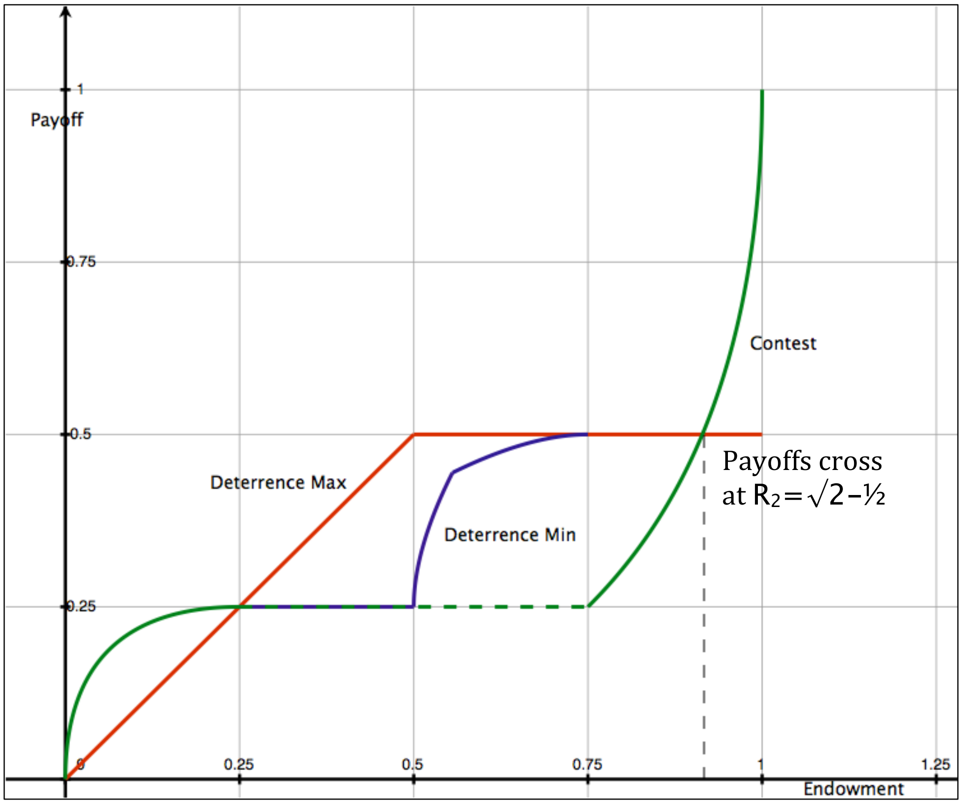

Figure 5: Comparison of payoffs under pure contest and full deterrence

full-deterrence. If initial endowments are outside these limits then in the equilibrium outcome there is war, and payoffs are equal to the pure contest payoffs for those endowments. If $R_1 = R_2 = \frac{1}{2}$*, then there are both full-deterrence equilibria and a war equilibrium, and all of the full-deterrence equilibria pareto-dominate the war equilibrium.*

Each equilibrium is pareto-optimal, since under minimal full deterrence the contributions of the two players are (imperfect) substitutes for each other. However, for a given initial distribution of resources, the total consumption in the economy in an equilibrum differs with the allocation of contributions between the two players. A possible measure of aggregate efficiency is total consumption in the economy:

$$c = 1 - g - x.$$

We can compute c in the pure conflict outcome corresponding to each distribution of resources. In full deterrence equilibria $x = 0$, so $c = 1 - g$, hence the most efficient equilibrium is the one that minimizes g. But since $g = \hat{g}(\min\{w_1, w_2\})$, this is equivalent to maximizing $\min\{w_1, w_2\}$. This can be restated as:

Proposition 5 *For resource distributions that accommodate multiple full deterrence equilibria, the Rawlsian criterion provides the most efficient allocation of public defense contributions.*

The proof follows directly from Lemma 1, and is omitted.

Proposition 5 says that, for efficient full-deterrence, the richer agent must make the maximum contribution consistent with full deterrence. If contributions were allocated as taxes by a public authority, then Proposition 5 leads to the following:

Corollary 3 *Suppose that when full deterrence is mutually incentive compatible, a public authority raises public defense contributions through taxes. Then the most efficient taxation scheme is one that is most progressive subject to incentive-compatibility.*

The efficient contributions and consumption profiles corresponding to full-deterrence equilibria can be computed from Proposition 3 for different resource distributions , and are as follows:

- For $\frac{4}{9} < R_1, R_2 < \frac{5}{9}$ the efficient equilibrium outcome is $w_1 = w_2 = \frac{4}{9},\ g = \frac{1}{9}$.
- For $\frac{5}{9} \leq R_2 \leq \frac{3}{4}$ the efficient equilibrium outcome is $w_1 = R_1,\ g = (1 - \sqrt{w_1})^2,\ w_2 = R_2 - g$.
- For $\frac{3}{4} \leq R_2 \leq \sqrt{2} - \dfrac{1}{2}$, the unique equilibrium outcome is $w_1 = R_1,\ g = \dfrac{1}{2} - R_1,\ w_2 = \dfrac{1}{2}$.

Finally we note that full deterrence is not efficient over the entire range in which it is an equilibrium. there is a range to the left of $R_2 = \sqrt{2} - \dfrac{1}{2}$ where the pure contest outcome is more efficient than the equilibrium outcome, but the equilibrium is full deterrence.

Proposition 6 *In the range $R_1 \in (\frac{3}{2} - \sqrt{2}, 1 - \frac{\sqrt{3}}{2})$, the equilibrium is full deterrence where conflict would yield a more efficient outcome.*

The intuition is that in this range the richer player unilaterally pays for deterrence, and for him the deterrence payoff is larger than the conflict payoff. Hence he unilaterally ensures full-deterrence. However, the poorer player would gain relative to his initial endowment in a pure contest, and this gain is larger than the loss that the richer player would suffer if pure contest replaced the equilibrium deterrence outcome.

Thus when income distributions are very unequal (but not sufficiently unequal for public defense to become non-viable) the equilibrium outcome is deterrence through public defense, but this is inefficient. In other words, for resource distributions in this range, the government is too large.

6. An extension

Throughout this paper I assumed that, if there is conflict, then the public defender sides against the aggressor, and if the aggressor is defeated it turns over the spoils of war to the agressee. However, peacekeepers or defenders in reality, be they governments or international organizations, do not act in this way. Some or all of the agressor's wealth may be confiscated, and he may be subjected to penalties or sanctions. Proceeds may be used to make reparations to the other party. However, the peacekeeper does not simply reduce to a mercenary army at the service of the agent that has been attacked.

Fortunately the model is not sensitive to a modification of this assumption. Consider the following alternative specification of payoffs:

1. If there is no conflict then each agent consumes any resources that remain after contributions and arms expenditures as before.
2. If there is conflict and there is a single aggressor (i.e., $a_1+a_2 = 1$), then any public defence contributions are pooled with the arms of the agent that is attacked. If the aggressor loses, then his remaining wealth is confiscated by the public defender and destroyed. In particular, the agent that was attacked does not keep the aggressor's remaining resources.

Note that this does not change the agressor's incentives, since if he is defeated then he loses his remaining resources anyway, and his payoff is no different if he wins. For the other agent, it is now less attractive to contribute to public defence if he is anticipating a conflict.

However, he would anticipate a conflict only if his contribution is less than sufficient to ensure full-deterrence. In this case he will not contribute at all, but instead channel the corresponding resources to private arms. In other words, with this alternative specification of payoffs, Assumption 1 becomes a proposition. Since I have used that assumption throughout, the earlier results all continue to hold.

Define the game Γ' as the game form described in Section 3.1.1 with the payoff function Π' described below:

- If $(a_1, a_2) = (0, 0)$, then

$$\Pi'_i(z) = R_i - g_i - x_i, \quad i = 1, 2.$$

- If $(a_1, a_2) = (1, 1)$ then

$$\Pi'_i(z) = \frac{x_i}{x_i + x_j}[1 - x - g]$$

- If $a_i = 1$ and $a_j = 0$, then

$$\begin{aligned} \Pi'_i(z) &= \frac{x_i}{x_i + x_j + g}[1 - x - g] \\ \Pi'_j(z) &= \frac{x_j + g}{x_i + x_j + g}[R_j - g_j - x_j] \end{aligned}$$

Note that the only difference between the two payoff functions is in j's payoff when $a_i = 1$ and $a_j = 0$.

Proposition 7 *: z^* is an equilibrium of Γ' if and only if it is an equilibrium of Γ.*

(This needs a proof! I am sure of the "if", but not yet happy with the proof of the "only if".)

7. Conclusion

This paper re-examines a standard model of contest in anarchy, where two agents possess resources that can be devoted to consumption or to acquisitive warfare. In the simplest version of that economy, the equilibrium necessarily involves conflict. However, since war is wasteful, it is likely that one or both agents would be willing to precommit to avoid conflict, even if such precommitment is somewhat costly.

This is a context that is intuitively conducive to the genesis of a peacekeeping state. I use a simple model of an exogenous peacekeeper which is effective to the extent that it is endowed with resources voluntarily provided by the potential contestants. I show that, for a large range of distributions of income, the agents will voluntarily commit sufficient resources to the peacekeeper. The resultant equilibrium is characterised by the absence of conflict.

However, when inequality is extreme, the peacekeeper is not endowed in equilibrium, and there is conflict. For less extreme but still high inequality, the peacekeeper is endowed and there is peace, but this is less efficient than pure conflict.

According to this paper, when inequality is low to moderate (in a sense made precise), all agents find that the existence of a peacekeeping government is in their interest. For higher inequality, the poorer agent finds government contrary to his interest, and at very high levels of inequality a peacekeeping government is incompatible with the interests of either agent. We should expect to see the least conflict in more equal societies and the most in very unequal ones, which is an observation that is possibly consistent with casual empiricism.

However, it is important to note that the political assumptions underlying this paper are extremely naive. I have assumed throughout that the public defender acts impartially, even though it may be funded entirely or largely by the richer agent. If instead the agent that contributes more to the defender can bend it to his own purposes, then the defender reduces to a militia, and clearly our peace equilibria will break down. Hence the analysis here needs to be supported by a much more sophisticated political theory of the nature of the state. However, that analysis is beyond the scope of the present paper.

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Appendix: Proofs

Proof of Lemma 1.
Let g be given and set $x_j = 0$. If i attacks with arms x_i, then his expected payoff is

$$\Pi_i(x_i, g) \;=\; \frac{x_i}{x_i + g}[1 - (x_i + g)].$$

This is maximised at $\tilde{x}_i(g) = \sqrt{g} - g$, and is increasing in x_i for $x_i < \sqrt{g} - g$ Thus i will choose to arm up to $x_i(g, w_i)$ where

$$x_i(g, w_i) = \begin{cases} \sqrt{g} - g & \text{if} \quad w_i \geq \sqrt{g} - g \\ w_i & \text{if} \quad w_i < \sqrt{g} - g \end{cases}$$

His corresponding payoff can be obtained from the previous two expressions:

$$\Pi_i(g, w_i) = \begin{cases} (1 - \sqrt{g})^2 & \text{if} \quad w_i \geq \sqrt{g} - g \\ \frac{w_i}{w_i + g}[1 - (w_i + g)] & \text{if} \quad w_i < \sqrt{g} - g \end{cases}$$

Define $g(w_i)$ as the value of g that solves $w_i = \Pi_i(g, w_i)$. Since $\Pi_i(g, w_i)$ is decreasing in g, it follows that g deters i if and only if $g \geq g(w_i)$.
Now let $g = g(w_i)$.
First suppose $w_i \geq \sqrt{g} - g$, so $w_i = \Pi_i(g, w_i) = (1 - \sqrt{g})^2$. Then $(1 - \sqrt{g})^2 \geq \sqrt{g} - g \;\Rightarrow\; g \leq \frac{1}{4} \Rightarrow w_i \geq \frac{1}{4}$. Conversely note that $\sqrt{g} - g \leq \frac{1}{4} \;\; \forall g \in [0, 1]$, so if $w_i \geq \frac{1}{4}$ then $\Pi_i(g, w_i) = (1 - \sqrt{g})^2$. Hence it follows that if $w_i \geq \frac{1}{4}$ then $g(w_i) = (1 - \sqrt{w_i})^2$.
Next let $w_i < \sqrt{g} - g$ so $w_i = \frac{w_i}{w_i + g}[1 - (w_i + g)] \quad \Rightarrow \quad g = \dfrac{1}{2} - w_i$. Then $w_i < \sqrt{\dfrac{1}{2} - w_i} - (\dfrac{1}{2} - w_i)$ *Rightarrow* $w_i < \frac{1}{4}$. Conversely, if $w_i < \frac{1}{4}$ then by the previous paragraph we cannot have $(1 - \sqrt{g})^2 \geq \sqrt{g} - g$, hence indeed $w_i < \sqrt{g} - g$.

from the above it follows that:

$$g(w_i) = \begin{cases} (1-\sqrt{w_i})^2 & \text{if} \quad w_i \geq \frac{1}{4} \\ \dfrac{1}{2} - w_i & \text{if} \quad w_i < \frac{1}{4} \end{cases}$$

Finally, suppose $w_j \geq w_i$. If $w_i \geq \frac{1}{4}$ then $g(w_i) = (1-\sqrt{w_i})^2 > (1-\sqrt{w_j})^2 = g(w_j)$. If $w_i < \frac{1}{4}$ and $g(w_i) = \dfrac{1}{2} - w_i > \frac{1}{4}$ then $w_j = 1 - w_i - g(w_i) = \dfrac{1}{2}$ and $g(w_j) = (1 - \frac{1}{\sqrt{2}} < \frac{1}{4} < g(w_i)$. Hence if $w_j \geq w_i$, then $g(w_i)$ is sufficient to deter j.
This establishes the lemma. ■

Proof of Lemma 2:
(i) If $a_i = 1$, then $j \neq i$ improves his payoff by setting $a_j = 0$ since then in the subsequent contest the public defence is added to j's arms.
(ii) Any increase in g reduces i's payoff if i will attack, hence it is suboptimal for him to contribute. ■

Proof of Proposition 1:
(i) If $\mathbf{g}$ is full deterrent, then neither player has an incentive to arm and attack if the other does not. Hence in the subgame $\mathbf{x}^* = 0$ and $\mathbf{a}^* = 0$ is an equilibrium. Further if $g > \hat{g}(\mathbf{w})$ and there is peace in the subgame, then at least one player i can increase his payoff by reducing g_i, so in equilibrium $g = \hat{g}(\mathbf{w})$.
(ii) Let $a_1^* = 1$, then by Lemma 2 $a_2^* = 0$, $g_1^* = 0 \;\Rightarrow\; g^* = g_2^*$, and $x_1^* > 0$. Since z^* is an equilibrium, it must be true that x_1^* is the optimal pure-contest response to $g_2^* + x_2^*$ given R_1, and $g_2^* + x_2^*$ is the optimal pure-contest response to x_1^* given R_2. Hence each player acquires exactly the arms he would acquire in the pure context equilibrium given (R_1, R_2). ■

Proof of Proposition 2
Corresponding to Lemma 1 consider the two cases of minimal full deterrence:
(i) If $w_1 < \frac{1}{4}$ then $\hat{g}(w_1) = \dfrac{1}{2} - w_1$ and $w_2 = \dfrac{1}{2}$.
(ii)If $\min\{w_1, w_2\} = w_1 \geq \frac{1}{4}$. Then $\hat{g}(w_1) = (1-\sqrt{w_1})^2$. Hence

$$w_2 = 1 - [w_1 + (1-\sqrt{w_1})^2] = 2(\sqrt{w_1} - w_1)$$

It can be checked that $w_2 \geq w_1$ provided $w_1 \leq \frac{4}{9}$. When $w_1 = \frac{4}{9}$, we have $w_1 = w_2$, and $\hat{g}(w_1) = \frac{1}{9}$.
We know that $hatg(w)$ depends only on $\min\{w_1, w_2\}$ ad $g > \hat{g}(w)$ is also full deterrent. Hence all values of w_2 between w_1 and the value derived above are consistent with full deterrence.
If $w_2 \geq w_1 > \frac{1}{4}$, then from (ii) above $w_1 + g(w_1) + w_2 \geq 2w_1 + (1-\sqrt{w_1})^2 = 3w_1 - 2\sqrt{w_1} + 1$. But the last term is ≤ 1 only if $w_1 \leq \frac{4}{9}$. Hence full deterrence is not feasible with $\min\{w_1, w_2\} > \frac{4}{9}$. ■

Proof of Proposition 3.

We focus on the case $R_1 \leq R_2$. The proof for the complementary case is symmetrical.

First suppose that $R_1 < \frac{1}{4}$. Then the post-contribution allocation must have $\min\{w_1, w_2\} < \frac{1}{4}$, and hence full deterrence requires $g = \frac{1}{2} - \min\{w_1, w_2\}$. Hence contributions by i s.t. $w_i = \min\{w_1, w_2\}$ do not alter the contribution required by $j \neq i$, so the only incentive compatible contribution from i is $g_i = 0$, and j must contribute $g_j = R_j - \frac{1}{2}$. It follows that when $R_1 < \frac{1}{4}$, the only contribution profile that is a candidate for equilibrium is $(g_1, g_2) = (0, R_2 - \frac{1}{2})$, which yields the consumption profile $(R_1, \frac{1}{2})$.

Next consider $\frac{1}{4} \leq R_1 \leq \frac{1}{2} \leq R_2$. For each R_1 In this range there are multiple configurations $\mathbf{g}$ that are consistent with minimal full deterrence. Recall that in this case the equilibrium conflict payoff for each player is $\frac{1}{4}$, which is the upper bound on the payoff that either player can attain in subgame Γ_2 if public contributions in stage 1 do not attain full deterrence. It therefore follows that the maximum contribution i is willing to make is $g_i \leq (R_i - \frac{1}{4})$.

First consider $R_1 \in [\frac{1}{4}, \frac{4}{9}] \;\Rightarrow\; R_2 \in [\frac{5}{9}, \frac{1}{4}]$. We know from Lemma 2 that player 2 can unilaterally ensure full deterrence by contributing $(1 - \sqrt{R_1})^2$, which leaves him with consumption $w_2 = 2(\sqrt{R_1} - R_1) \geq w_1$. Since 1 does not contribute, $w_1 = R_1 = 1 - R_2$, the resultant consumption vector is $(R_1, 2[\sqrt{1 - R_2} - (1 - R_2)])$.

For $R_1 \in (\frac{4}{9}, \frac{1}{2}]$, if player 2 contributes sufficiently to deter player 1, this leaves him with $w_2 < w_1$. Hence to ensure full deterrence with no contribution from player 1, he must deter himself. This implies $g_2 = (1 - \sqrt{w_2})^2$. Since $g_2 + w_2 = R_2$, This leaves player 2 a consumption of $[\frac{1}{2}\{1 + \sqrt{(2R_2 - 1)}\}]^2$, which ranges from $w_2 = \frac{4}{9}$ when $R_2 = \frac{5}{9}$ to $w_2 = \frac{1}{4}$ when $R_2 = \frac{1}{2}$.

The payoffs for the complementary range can be found symmetrically. ■

Proof of Proposition 4

We focus on $\mathbf{R} : R_1 \leq R_2$. An identical proof applies when the inequality is reversed. We consider three cases.

Case 1: $R_1 \in [\frac{1}{4}, \dfrac{1}{2}) \;\Leftrightarrow\; R_2 \in (\dfrac{1}{2}, \frac{3}{4}]$.

Suppose there is an equilibrium z^* such that $\mathbf{a}^* \neq 0$, i.e., there is war. Then $\mathbf{g}^*$ is not full deterrent, and by Proposition 1 the players receive the pure contest payoffs $(\frac{1}{4}, \frac{1}{4})$. Thus player 1 will never contribute more than $R_1 - \frac{1}{4}$ to public defence, hence in any equilibrium we must have $w_1 \geq \frac{1}{4} = \Pi_1^C$.

But then the largest contribution 2 must make to ensure full-deterrence is $g_2 = \frac{1}{4} - g_1 \leq \frac{1}{4}$, which leaves him with $w_2 > \frac{1}{4} = \Pi_2^C$. Hence for any incentive compatible contribution from player 1, player 2 prefers to ensure full-deterrence rather than engage in contest. Thus if there is an equilibrium then it must be full-deterrence.

It is easy to verify that $\mathbf{g} = (R_1 - \frac{1}{4}, \dfrac{1}{2} - R_1)$ is an equilibrium. Hence there is at least one equilibrium, and any equilibrium is full-deterrence.

Case 2: $R_1 = R_2 = \dfrac{1}{2}$.

The arguments for case 1 carry over for any contribution $0 < g_1 < \frac{1}{4}$, which lead to full-deterrence equilibria with payoffs $\Pi_i > \frac{1}{4},\ i = 1, 2$.

However, for $g_1 = 0$, player 2 has two optimal choices; he can contribute $g_2 = \frac{1}{4}$, which ensures full-deterrence and yields him a payoff of $\frac{1}{4}$, or he can set $g_2 = 0$ (cf. Assumption 1), leading to war which also yields a payoff of $\frac{1}{4}$. An equivalent argument applies to player 1, hence $\mathbf{g} = (0, 0)$ followed by a pure contest is an equilibrium.
It follows that the deterrence equilibria where each player contributes a strictly positive to public defence strictly dominate the unique pure contest equilibrium. Further, if $g_1 = 0$, the deterrence equilibrium with $g_2 = \frac{1}{4}$ yields payoffs $(\dfrac{1}{2}, \frac{1}{4})$, which pareto dominates the contest outcome.

Case 3: $R_2 > \frac{3}{4} \Leftrightarrow R_1 < \frac{1}{4}$.
By Proposition 3, any full-deterrence equilibrium must have $g_1 = 0$ and $g_2 = R_2 - \dfrac{1}{2}$, yielding player 2 a payoff of $\dfrac{1}{2}$.
A pure contest equilibrium yields player 2 a payoff of $(1 - \sqrt{R_1})^2$.
Hence the nature of the equilibrium depends on player 2's choices, and he will choose to deter if and only if

$$\frac{1}{2} \geq (1 - \sqrt{R_1})^2 \quad \Leftrightarrow \quad R_1 \geq (\frac{3}{2} - \sqrt{2}).$$

■

Proof of Proposition 6

Let $\mathbf{R} : R_1 \in [\frac{3}{2} - \sqrt{2}, \frac{1}{4}]$. We know from an earlier proposition that in this range player 2 unilaterally ensures full-deterrence in equilibrium, with $g_2 = \dfrac{1}{2} - R_1$. Hence the sum of consumptions in equilibrium is $c^* = \dfrac{1}{2} + R_1$.

Consider the corresponding pure contest outcome (see Section **??**). Since player 1 is constrained, she invests her entire endowment in arms, and 2 responds optimally, which yields payoffs $\Pi_1 = \sqrt{R_1}(1 - \sqrt{R_1})$ and $\Pi_2 = (1 - \sqrt{R_1})^2$. Thus total consumption is $c^C = (1 - \sqrt{R_1})$.

Thus full-deterrence is efficient if and only if

$$\frac{1}{2} + R_1 \geq (1 - \sqrt{R_1}),$$

which reduces to $R_1 \geq 1 - \frac{\sqrt{3}}{2}$, which it can be verified is greater than $\frac{3}{2} - \sqrt{2}$. ■